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# ON LOCALLY UNIT REGULARITY CONDITIONS FOR ARBITRARY LEAVITT PATH ALGEBRAS

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Abstract. Let $\Gamma$ be a graph, $K$ be any field and $S$ be the endomorphism ring of $L := L_K(\Gamma)$ considered as a right $L$-module. In this paper, we give defination of the left locally unit regular ring. We show that (1) if $S$ is locally unit regular, then $L$ is locally unit regular, (2) if $L$ is morphic and image projective then $S$ is left morphic, (3) $S$ is a directly finite ring then $L$ is directly finite, (4) if $S$ is an exchange ring then $L$ is directly finite and if $L$ is a direct finite ring then $L$ is an exchange ring.

## 1. Introduction

Throughout this article $\Gamma$ will denote a directed graph, $K$ will denote an arbitrary field and the Leavitt path algebras (shortly LPAs) of $\Gamma$ with coefficients in $K$ will denoted $L := L_K(\Gamma)$.

LPAs can be regarded as the algebraic counterparts of the graph $C^*$-algebras, the descendants from the algebras investigated by Cuntz in [6]. LPAs can be viewed as a broad generalization of the algebras constructed by Leavitt in [11] to produce rings without the Invariant Basis Number property. LPAs associated to directed graphs were introduced in [4, 1]. These $L_K(\Gamma)$ are algebras associated to directed graphs and are the algebraic analogs of the Cuntz-Krieger graph $C^*$-algebras [15].

Let $\Gamma$ be a graph and $K$ a field. In [3], G. Abrams and K. M. Rangaswamy showed how definition of von Neumann regular ring (recall that a ring $R$ is von Neumann regular if for every $a \in R$ there exists $b \in R$ such that $a = aba$) is extended to locally unit regular ring and in [3, Theorem 2] if $\Gamma$ is arbitrary graph, $L_K(\Gamma)$ is locally unit regular if and only if $\Gamma$ is acyclic. This article is organized as follows. In Section 2, we recall some preliminaries about LPAs which we need in the next section. In Section 3, for the ring $S$ of endomorphism ring of $L_K(\Gamma)$ (viewed as a right $L_K(\Gamma)$-module), we prove that: (1) if $S$ is locally unit regular,

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then $L$ is locally unit regular, (2) if $L$ is morphic and image projective then $S$ is left morphic, (3) if $S$ is a directly finite ring then $L$ is directly finite, (4) if $S$ is an exchange ring then $L$ is directly finite and if $L$ is a direct finite ring then $L$ is an exchange ring.

## 2. Definitions and Preliminaries

We recall some graph-theoretic concepts, the definition and standard examples of LPAs.

**Definition 1.** *A (directed) graph $\Gamma = (V, E, r, s)$ consist of two set $V$ and $E$ (with no restriction on their cardinals) together with maps $r, s : E \to V$. The elements of $V$ are called vertices and the elements of $E$ edges. For $e \in E$, the vertices $s(e)$ and $r(e)$ are called the source and range of $e$. If $s^{-1}(v)$ is a finite set for every $v \in V$, then the graph is called row-finite. If $V$ is finite and $\Gamma$ is row finite, then $E$ must necessarily be finite as well; in this case we say simply that $\Gamma$ is finite.*

*A vertex which emits (receives) no edges is called a sink (source). A vertex $v$ is called an infinite emitter if $s^{-1}(v)$ is an infinite set. A vertex $v$ is a bifurcation if $s^{-1}(v)$ has at least two elements. A path $p$ in a graph $\Gamma$ is a finite sequence of edges $p = e_1 ... e_n$ such that $r(e_i) = s(e_{i+1})$ for $1 \leq i \leq n-1$. In this case, $s(p) = s(e_1)$ and $r(p) = r(e_n)$ are the source and range of $p$, respectively, and $n$ is the length of $p$. We view the elements of $V$ as paths of length 0.*

*A path $p = e_1 ... e_n$ is said to be closed path based at $v$ if $s(p) = v = r(p)$. If $p$ is an closed path in $\Gamma$ and $s(e_i) \neq s(e_i)$ for all $i \neq j$, then $p$ is said to be a cycle. A cycle consisting of just one edge is called a loop. A graph which contains no cycles is called acyclic. A graph $\Gamma$ is said to be no-exit if no vertex of any cycle is a bifurcation.*

**Definition 2.** *(LPAs of Arbitrary Graph)*

*For an arbitrary graph $\Gamma$ and a field $K$, the Leavitt path $K$-algebra of $\Gamma$, denoted by $L_K(\Gamma)$, is the $K$-algebra generated by the set $V \cup E \cup \{e^* | e \in E\}$ with the following relations,*

1. *$v_i v_j = \delta_{v_i, v_j} v_i$ for every $v_i, v_j \in V$*
2. *$s(e)e = e = er(e)$ for all $e \in E$.*
3. *$r(e)e^* = e^* = e^* s(e)$ for all $e \in E$.*
4. *(CK1) $e^* f = \delta_{e,f} r(e)$ for all $e, f \in E$.*
5. *(CK2)$v = \sum_{\{e \in E, s(e) = v\}} ee^*$ for every $v \in V$ that is neither a sink nor an infinite emitter.*

The first three relations are the path algebra relations. The last two are the so-called Cuntz-Krieger relations. We let $r(e^*)$ denote $s(e)$, and we let $s(e^*)$ denote $r(e)$. If $p = e_1 ... e_n$ is a path in $\Gamma$, we write $p^*$ for the element $e_n^* ... e_1^*$ of $L_K(\Gamma)$. With this notation, the LPA $L_K(\Gamma)$ can be viewed as a $K-$vector space span of $\{pq^* \mid p, q \text{ are paths in } \Gamma\}$. (Recall that the elements of $V$ are viewed as paths of length 0, so that this set includes elements of the form $v$ with $v \in V$.)

If $\Gamma$ is a finite graph, then $L_K(\Gamma)$ is unital with $\sum_{v\in V} v = 1_{L_K(\Gamma)}$; otherwise, $L_K(\Gamma)$ is a ring with a set of local units consisting of sums of distinct vertices of the graph.

Many well-known algebras can be realized as the LPAs of a graph. The most basic graph configuration is shown below (the isomorphism for can be found in [1]).

**Example 1.** *The ring of Laurent polynomials $K[x, x^{-1}]$ is the LPA of the graph given by a single loop graph.*

We will now outline some easily derivable basic facts about the endomorphism ring $S$ of $L := L_K(\Gamma)$. Let $\Gamma$ be any graph and $K$ be any field. Denote by $S$ the unital ring $End(L_L)$. Then we may identify $L$ with subring of $S$, concretely, the following is a monomorphism of rings:

$$\begin{aligned} \phi : L &\to End(L_L) \\ x &\mapsto \lambda_x \end{aligned}$$

where $\lambda_x : L \to L$ is the left multiplication by $x$, i.e., for every $y \in L$, $\lambda_x(y) = xy$ which is a homomorphism of right $L-$module. The map $\phi$ is also a monomorphism because given a nonzero $x \in L$ there exists an idempotent $u \in L$ such that $xu = x$, hence $0 \neq x = \lambda_x(u)$.

## 3. Results

According to Abrams and Rangaswamy [3]:

• A (possibly nonunital) ring $R$ is called a *ring with local units* if, for each finite subset $S$ of $R$, there exists an idempotent $e$ of $R$ such that $S \subseteq eRe$;

• If $R$ is a ring with local units then $R$ is called *locally unit regular* if for each $a \in R$ there is an idempotent (a local unit) $v$ and local inverses $u, u'$ such that $uu' = v = u'u$, $va = a = av$ and $aua = a$ (see [3, Definition 6]).

**Theorem 1.** *Let $\Gamma$ be an arbitrary graph, $K$ be any field and $S$ be the endomorphism ring of $L := L_K(\Gamma)$.*

(1) *If $S$ is locally unit regular, then $L$ is locally unit regular. Moreover $L$ is regular.*
(2) *If $L$ is locally unit regular, then $vLv$ is locally unit regular for every non zero idempotent $v$ of $L$.*

*Proof.* (1) Take $x \in L$. Since $S$ is local unit regular, there exists an idempotent $e \in S$ such that $\lambda_x \in eSe$ and elements $f, g \in eSe$ such that $fg = e = gf$ and $\lambda_x f \lambda_x = \lambda_x$. Choose an idempotent $u \in L$ such that $x\lambda_{e(u)} = x = \lambda_{e(u)}x$ so $x \in \lambda_{e(u)} L \lambda_{e(u)}$. Note that,

$$\lambda_{f(u)}\lambda_{g(u)} = \lambda_{e(u)} = \lambda_{g(u)}\lambda_{f(u)}$$

and

$$\lambda_x = \lambda_x\lambda_{f(x)} = \lambda_x\lambda_{f(ux)} = \lambda_x\lambda_{f(u)}\lambda_x.$$

Since $f \in eSe$, there exists $h \in S$ such that $f = ehe$. Then $f(u) = e(u)h(u)e(u)$, so

$$\lambda_{f(u)} = \lambda_{e(u)h(u)e(u)} = \lambda_{e(u)}\lambda_{h(u)}\lambda_{e(u)}$$

and we get $\lambda_{f(u)} \in \lambda_{e(u)}L\lambda_{e(u)}$. Similarly $\lambda_{g(u)} \in \lambda_{e(u)}L\lambda_{e(u)}$. Hence $L$ is locally unit regular.

(2) Take any $a \in vLv$. Since $L$ is locally unit regular, there exist an idempotent $e$ and local inverses $u, u'$ such that $ea = a = ae$, $uu' = e = u'u$ and $aua = a$. As $ea = a$ and $av = a$ which imply $vea = va = a = ea$ and $eav = ea$ respectively, we get $ea = eav = vea$. Now $ea \in vLv$, which implies $e \in vLv$. Then $ve = e = ev$. Let $e^* = vev$, $h = vuv$ and $h' = veu'ev$. Note that

$$e^*e^* = (vev)(vev) = vevev = veevv = vev = e^* \in vLv$$

$$hh' = (vuv)(veu'ev) = vuveu'ev = vueu'ev = veuu'^*$$

$$h'h = (veu'ev)(vuv) = veu'evuv = veu'euv = veu'^*$$

$$aha = a(vuv)a = vauav = vav = a,$$

which imply $vLv$ is locally unit regular. □

**Definition 3.** *A ring $R$ is dependent if, for each $a, b \in R$, there are $s, t \in R$, not both zero, such that $sa + tb = 0$.*

Let $\Gamma$ be an arbitrary graph, $K$ be any field and $S$ be the endomorphism ring of $L := L_K(\Gamma)$ considered as a right $L$-module. If $S$ is dependent so is $L$. In fact, suppose $S$ is dependent and $a, b \in L$. Then there are elements $f, g \in S$, not both zero, such that $f\lambda_a + g\lambda_b = 0$. If $u_1$ and $u_2$ are local units in $L$ satisfying $u_1a = a = au_1$ and $u_2b = b = bu_2$, then

$$f\lambda_a = f\lambda_{u_1a} = f\lambda_{u_1}\lambda_a = \lambda_{f(u_1)}\lambda_a$$

and

$$g\lambda_b = g\lambda_{u_2b} = g\lambda_{u_2}\lambda_b = \lambda_{g(u_2)}\lambda_b.$$

Now

$$\begin{aligned} 0 &= f\lambda_a + g\lambda_b \\ &= \lambda_{f(u_1)}\lambda_a + \lambda_{g(u_2)}\lambda_b, \end{aligned}$$

and hence $L$ is dependent.

In the literature on von-Neumann regular rings, various conditions have been shown to characterize the subclass of unit regular rings. In [8, Theorem 6], Ehrlich showed that every unit regular ring $R$ is dependent. In [10, Corollary 10], Henriksen shows that not all dependent regular rings are unit regular. The following observation gives one more such condition for dependent rings.

**Theorem 2.** *If $L_K(\Gamma)$ is locally unit regular, then it is dependent.*

*Proof.* Let $L_K(\Gamma)$ be locally unit regular and let some elements provide locally unit regular condition in the definition. Take $a, b \in L_K(\Gamma)$. If both $a$ and $b$ have local inverses in $L_K(\Gamma)$, then there exist $u_1$ and $u_2$ in $L_K(\Gamma)$ such that $u_1 a = v$ and $u_2 b = v$ for local unit $v$ in $L_K(\Gamma)$. So, we get $sa + tb = 0$, where $s = u_1$ and $t = -u_2$. If one of the elements, say $a$, has no local inverse in $L_K(\Gamma)$, by definition of locally unit regularity, then we can write $aua = a \Rightarrow aua = va \Rightarrow (au - v)a = 0$. Now we get $au - v \neq 0$. Assume $au - v = 0$. So $au = v$, it is a contradiction. Then, for $s = (au - v) \neq 0$ and $t = 0$, which implies $sa + tb = 0$. □

**Definition 4.** *Let $R$ be a ring with local units. We call $R$ left (right) locally unit regular ring if for each $a \in R$ there exist an idempotent $v \in R$ and left (right) local inverses $u, u'$ such that $u'u = v$ ($uu' = v$), $va = a$ ($av = a$) and $aua = a$.*

**Definition 5.** *([12]) Let $M$ be a right $R$-module, and let $S = End_R(M)$. Then $M$ is called is a d-Rickart (or dual Rickart) module if the image in $M$ of any single element of $S$ is a direct summand of $M$. Clearly, $R_R$ a d-Rickart module iff $R$ is a regular ring.*

**Definition 6.** *Given paths $p, q \in \Gamma$, we say that $q$ is an initial segment of $p$ if $p = qm$ for some path $m \in \Gamma$. It is well known that, given non-zero paths $pq^*$ and $mn^*$ in $L_K(\Gamma)$, $q$ is an initial segment of $m$ if and only if $(pq^*)(mn^*) \neq 0$.*

**Theorem 3.** *Let $\Gamma$ be a graph, $K$ be any field and $S$ be the endomorphism ring of $L := L_K(\Gamma)$ considered as a right $L$-module. The following conditions are equivalent.*

(1) *$S$ is left locally unit regular.*
(2) *$S$ is regular and, for all paths $x, y \in L$, $Sx = Sy$ implies $x$ is an initial segment of $y$.*
(3) *$L$ is dual-Rickart and, for all paths $x, y \in L$, $Sx = Sy$ implies $x$ is an initial segment of $y$.*

*Proof.* (1) ⇒ (2) Assume that $S$ is left locally unit regular. Hence $S$ is regular and $L$ is left locally unit regular by Theorem 1. Let $x, y \in L$ be two paths. Then there exist an idempotent $v \in L$ and left local inverses $v_1, v_2 \in L$ such that $vy = y$, $v_2 v_1 = v$ and $y = y v_1 y$. If $Sx = Sy$, then $x = f(y)$ for some $f \in S$. Now $y = y v_1 y$ implies $f(y) = f(y v_1 y)$, and so $x = \underbrace{f(y v_1)}_{\in L} y$. Hence $x$ is an initial segment of $y$.

(2) ⇒ (3) This follows from [17, Corollary 3.2].

(3) ⇒ (1) Assume that $L$ is dual-Rickart. Then $f(L)$ is a direct summand of $L$, where $f \in S$. Let $e$ be an idempotent in $S$ with $f(L) = eL$. Let $x \in L$. Then there exists $y \in L$ such that $f(x) = e(y)$. Now

$$(ef)(x) = e(f(x)) = e(e(y)) = e(y) = f(x),$$

which implies $ef = f$. Let $h$ be the left inverse of $f$ and $g = fe$. Then $gh = e$ and $fhf = f$. □

**Definition 7.** *([13]) An endomorphism $\alpha$ of a module $M$ is called morphic if $M/M\alpha \cong \mathrm{Ker}(\alpha)$, equivalently there exists $\beta \in \mathrm{End}(M)$ such that $M\beta = \mathrm{Ker}(\alpha)$ and $\mathrm{Ker}(\beta) = M\alpha$ by [13, Lemma 1]. The module $M$ is called a morphic module if every endomorphism is morphic. If $R$ is a ring, an element $a$ in $R$ is called left morphic if right multiplication $\cdot a :_R R \to_R R$ is a morphic endomorphism, that is if $R/Ra \cong l(a)$. The ring itself is called a left morphic ring if every element is left morphic, that is if ${}_RR$ is a morphic module.*

Note that if $S$ is dependent then $L_K(\Gamma)$ is morphic by [14, Corollary 3.5].

**Theorem 4.** *Let $\Gamma$ be any graph and let $K$ be any field. If $L_K(\Gamma)$ is left morphic and regular ring then $L_K(\Gamma)$ is left locally unit regular ring.*

*Proof.* Let $L_K(\Gamma) = L$ be left morphic and regular ring. Then each $a \in L$ is both regular and morphic. So, there exist an $x \in L$ such that $a = axa$ and for some $b \in L$, $La = ann(b)$ and $Lb = ann(a)$. Let $u = xax + b$. Then $a = aua$. To see that $u$ is left local inverse, since $L$ has local units, choose an idempotent $v \in L$ such that $va = a$. Then we get, $0 = va - a = va - axa = (v - ax)a$, so $v - ax \in ann(a) = Lb$ and there exists an element $y \in L$ such that $v - ax = yb$. We take $u' = a + y(v - ax)$. We show that $u'u = v$:

$$\begin{aligned} u'u &= (a + y(v - ax))(xax + b) \\ &= axax + ab + y(v - ax)xax + y(v - ax)b \\ &= ax + ab + yvxax - yxaxax + yvb - yxab \\ &= ax + yb = v \end{aligned}$$

Hence $L = L_K(\Gamma)$ is left locally regular ring. □

**Theorem 5.** *Let $\Gamma$ be a graph, $K$ be any field and $S$ be the endomorphism ring of $L := L_K(\Gamma)$ considered as a right $L$-module. If $L_K(\Gamma)$ is morphic and image projective then $S$ is left morphic.*

*Proof.* Let $L := L_K(\Gamma)$ be morphic and image projective. Given any $\alpha \in S$, since $L$ is morphic, we may choose an $\beta \in S$ such that, $L\alpha = ker(\beta)$ and $L\beta = ker(\alpha)$. Since $\alpha\beta = 0$, $S\alpha \subset ann^S(\beta)$. Conversely, if $\gamma \in ann^S(\beta)$ then $\gamma\beta = 0$ so $L\gamma \subset ker(\beta) = L\alpha$ and hence $\gamma \in S\alpha$ because $L$ is image projective. Thus $S\alpha = ann^S(\beta)$. We may see $S\beta = ann^S(\alpha)$ in the same way. Hence $S$ is left morphic. □

**Definition 8.** *([16, Definition 4.1]) If $R$ is a ring with local units then $R$ is called directly finite if for each $x, y \in R$ there is an idempotent $u$ such that $xu = x = ux$ and $yu = y = uy$, we have that $xy = u$ implies $yx = u$.*

**Theorem 6.** *Let $\Gamma$ be a graph, $K$ be any field and $S$ be the endomorphism ring of $L := L_K(\Gamma)$ considered as a right $L$-module. If $S$ is a directly finite ring then $L_K(\Gamma)$ is directly finite.*

*Proof.* Take any $x, y$ in $L_K(\Gamma)$. Since $S$ is a direct finite ring, there is an idempotent $\varepsilon$ in $S$ such that $\lambda_x\varepsilon = \lambda_x = \varepsilon\lambda_x$ and $\lambda_y\varepsilon = \lambda_y = \varepsilon\lambda_y$, we have that $\lambda_x\lambda_y = \varepsilon$ implies $\lambda_y\lambda_x = \varepsilon$. For an idempotent $u$ with $xu = x = ux$ and $yu = y = uy$,

$$\lambda_x\lambda_y = \varepsilon \Rightarrow \lambda_x\lambda_y\lambda_x = \varepsilon\lambda_x \Rightarrow \lambda_x = \varepsilon\lambda_{uv} \Rightarrow \lambda_x = \lambda_{\varepsilon(u)}\lambda_x$$

$$\lambda_x = \varepsilon\lambda_x = \lambda_{\varepsilon(x)} = \lambda_{\varepsilon(xu)} = \lambda_{\varepsilon(x)}\lambda_{\varepsilon(u)} = \varepsilon\lambda_x\lambda_{\varepsilon(u)}$$

So, $\lambda_x\lambda_{\varepsilon(u)} = \lambda_x = \lambda_{\varepsilon(u)}\lambda_x$. Similarly $\lambda_y\lambda_{\varepsilon(u)} = \lambda_y = \lambda_{\varepsilon(u)}\lambda_y$. Assume that, $\lambda_x\lambda_y = \lambda_{\varepsilon(u)}$. We then see that $\lambda_y\lambda_x = \lambda_{\varepsilon(u)}$.

$$\lambda_y\lambda_x = \lambda_y\lambda_{\varepsilon(u)}\lambda_x = \lambda_y\lambda_x\lambda_{\varepsilon(u)} = \varepsilon\lambda_{\varepsilon(u)} = \lambda_{\varepsilon^2(u)} = \lambda_{\varepsilon(u)},$$

as desired. □

Ones hopes that if $L_K(\Gamma)$ is directly finite then $L_K(\Gamma)$ is locally unit regular but this is not true. Because $K[x, x^{-1}]$ is a commutative Leavitt path algebra (of the graph with one vertex and one loop) clearly directly finite. But it is not von Neumann regular ring.

**Corollary 1.** *Let $\Gamma$ be a graph, $K$ be any field and $S$ be the endomorphism ring of $L := L_K(\Gamma)$ considered as a right $L$-module. If $S$ is a directly finite ring, then $\Gamma$ is no exit.*

*Proof.* Let $S$ be a directly finite ring. Then $L_K(\Gamma)$ is a directly finite ring. So, by [16, Proposition 4.3], $\Gamma$ is no exit. □

**Definition 9.** *$R$ is said to be a (left) exchange ring if for any direct decomposition $A = M \oplus N = \oplus_{i\in I} A_i$ of any left $R$-module $A$, where $R \cong M$ as left $R$-modules and $I$ is a finite set, there always exist submodules $B_i$ of $A_i$ such that $A = M \oplus (\oplus_{i\in I} B_i)$.*

**Theorem 7.** *Let $\Gamma$ be an infinite graph, $K$ be any field and $S$ be the endomorphism ring of $L := L_K(\Gamma)$ considered as a right $L$-module. Then*

(1) *If $S$ is an exchange ring then $L$ is directly finite.*
(2) *If $L$ is a direct finite ring then $L$ is an exchange ring.*

*Proof.* (1) Let $S$ be an exchange ring. Then, by [5, Proposition 2.10], $L_K(\Gamma)$ is an exchange ring. For every $x, y \in L$ and an idempotent $u \in L$ such that $xu = x = ux$ and $yu = y = uy$ we have that $xy = u$. We show that $yx = u$. Since $L$ is an exchange ring, there exist $r, s \in L$ such that $u = rx = s + x - sx$. So, $u = rx \Rightarrow uy = rxy \Rightarrow y = ru \Rightarrow yx = rux = rx = u$, as desired.

(2) Let $L$ be a direct finite ring. For any $x, y \in L$ and an idempotent $u \in L$ such that $xu = x = ux$ and $yu = y = uy$ we have that $xy = u$ implies $yx = u$. We show that $L$ is an exchange ring. For any $x \in L$ taking $r = y$ and $s = u$, we get $u = rx = s + x - sx$. So, $L$ is an exchange ring. □

**Corollary 2.** *Let $\Gamma$ be infinite graph, $K$ be any field and $S$ be the endomorphism ring of $L := L_K(\Gamma)$ considered as a right $L$-module. Then the following conditions are equivalent.*

(1) *$S$ is an exchange ring.*
(2) *$L_K(\Gamma)$ is an exchange ring.*
(3) *$L_K(\Gamma)$ is a directly finite ring.*
(4) *$E$ is no exit*

*Proof.* (1) $\Leftrightarrow$ (2) This is [5, Proposition 2.10].
(2) $\Leftrightarrow$ (3) This follows from Theorem 7 (1) and Theorem 7 (2).
(3) $\Leftrightarrow$ (4) This is [16, Teorem 4.12]. □

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