



## ON LOCALLY UNIT REGULARITY CONDITIONS FOR ARBITRARY LEAVITT PATH ALGEBRAS

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**ABSTRACT.** Let  $\Gamma$  be a graph,  $K$  be any field and  $S$  be the endomorphism ring of  $L := L_K(\Gamma)$  considered as a right  $L$ -module. In this paper, we give definition of the left locally unit regular ring. We show that (1) if  $S$  is locally unit regular, then  $L$  is locally unit regular, (2) if  $L$  is morphic and image projective then  $S$  is left morphic, (3)  $S$  is a directly finite ring then  $L$  is directly finite, (4) if  $S$  is an exchange ring then  $L$  is directly finite and if  $L$  is a direct finite ring then  $L$  is an exchange ring.

### 1. INTRODUCTION

Throughout this article  $\Gamma$  will denote a directed graph,  $K$  will denote an arbitrary field and the Leavitt path algebras (shortly LPAs) of  $\Gamma$  with coefficients in  $K$  will denoted  $L := L_K(\Gamma)$ .

LPAs can be regarded as the algebraic counterparts of the graph  $C^*$ -algebras, the descendants from the algebras investigated by Cuntz in [6]. LPAs can be viewed as a broad generalization of the algebras constructed by Leavitt in [11] to produce rings without the Invariant Basis Number property. LPAs associated to directed graphs were introduced in [4, 1]. These  $L_K(\Gamma)$  are algebras associated to directed graphs and are the algebraic analogs of the Cuntz-Krieger graph  $C^*$ -algebras [15].

Let  $\Gamma$  be a graph and  $K$  a field. In [3], G. Abrams and K. M. Rangaswamy showed how definition of von Neumann regular ring (recall that a ring  $R$  is von Neumann regular if for every  $a \in R$  there exists  $b \in R$  such that  $a = aba$ ) is extended to locally unit regular ring and in [3, Theorem 2] if  $\Gamma$  is arbitrary graph,  $L_K(\Gamma)$  is locally unit regular if and only if  $\Gamma$  is acyclic. This article is organized as follows. In Section 2, we recall some preliminaries about LPAs which we need in the next section. In Section 3, for the ring  $S$  of endomorphism ring of  $L_K(\Gamma)$  (viewed as a right  $L_K(\Gamma)$ -module), we prove that: (1) if  $S$  is locally unit regular,

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Received by the editors: November 25, 2016, Accepted: May 05, 2017.

2010 *Mathematics Subject Classification.* Primary 16D50, 16E50; Secondary 16U60, 16W20.

*Key words and phrases.* Leavitt path algebra, von Neumann regular ring, locally regular ring, endomorphism ring.

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Communications Faculty of Sciences University of Ankara-Series A1 Mathematics and Statistics.  
Communications de la Faculté des Sciences de l'Université d'Ankara. Série A1. Mathématiques et Statistiques.

then  $L$  is locally unit regular, (2) if  $L$  is morp hic and image projective then  $S$  is left morp hic, (3) if  $S$  is a directly finite ring then  $L$  is directly finite, (4) if  $S$  is an exchange ring then  $L$  is directly finite and if  $L$  is a direct finite ring then  $L$  is an exchange ring.

## 2. DEFINITIONS AND PRELIMINARIES

We recall some graph-theoretic concepts, the definition and standard examples of LPAs.

**Definition 1.** A (directed) graph  $\Gamma = (V, E, r, s)$  consist of two set  $V$  and  $E$  (with no restriction on their cardinals) together with maps  $r, s : E \rightarrow V$ . The elements of  $V$  are called vertices and the elements of  $E$  edges. For  $e \in E$ , the vertices  $s(e)$  and  $r(e)$  are called the source and range of  $e$ . If  $s^{-1}(v)$  is a finite set for every  $v \in V$ , then the graph is called row-finite. If  $V$  is finite and  $\Gamma$  is row finite, then  $E$  must necessarily be finite as well; in this case we say simply that  $\Gamma$  is finite.

A vertex which emits (receives) no edges is called a sink (source). A vertex  $v$  is called an infinite emitter if  $s^{-1}(v)$  is an infinite set. A vertex  $v$  is a bifurcation if  $s^{-1}(v)$  has at least two elements. A path  $p$  in a graph  $\Gamma$  is a finite sequence of edges  $p = e_1 \dots e_n$  such that  $r(e_i) = s(e_{i+1})$  for  $1 \leq i \leq n-1$ . In this case,  $s(p) = s(e_1)$  and  $r(p) = r(e_n)$  are the source and range of  $p$ , respectively, and  $n$  is the length of  $p$ . We view the elements of  $V$  as paths of length 0.

A path  $p = e_1 \dots e_n$  is said to be closed path based at  $v$  if  $s(p) = v = r(p)$ . If  $p$  is an closed path in  $\Gamma$  and  $s(e_i) \neq s(e_j)$  for all  $i \neq j$ , then  $p$  is said to be a cycle. A cycle consisting of just one edge is called a loop. A graph which contains no cycles is called acyclic. A graph  $\Gamma$  is said to be no-exit if no vertex of any cycle is a bifurcation.

**Definition 2.** (LPAs of Arbitrary Graph)

For an arbitrary graph  $\Gamma$  and a field  $K$ , the Leavitt path  $K$ -algebra of  $\Gamma$ , denoted by  $L_K(\Gamma)$ , is the  $K$ -algebra generated by the set  $V \cup E \cup \{e^* \mid e \in E\}$  with the following relations,

- (1)  $v_i v_j = \delta_{v_i, v_j} v_i$  for every  $v_i, v_j \in V$
- (2)  $s(e)e = e = er(e)$  for all  $e \in E$ .
- (3)  $r(e)e^* = e^* = e^*s(e)$  for all  $e \in E$ .
- (4) (CK1)  $e^*f = \delta_{e, f} r(e)$  for all  $e, f \in E$ .
- (5) (CK2)  $v = \sum_{\{e \in E, s(e)=v\}} ee^*$  for every  $v \in V$  that is neither a sink nor an infinite emitter.

The first three relations are the path algebra relations. The last two are the so-called Cuntz-Krieger relations. We let  $r(e^*)$  denote  $s(e)$ , and we let  $s(e^*)$  denote  $r(e)$ . If  $p = e_1 \dots e_n$  is a path in  $\Gamma$ , we write  $p^*$  for the element  $e_n^* \dots e_1^*$  of  $L_K(\Gamma)$ . With this notation, the LPA  $L_K(\Gamma)$  can be viewed as a  $K$ -vector space span of  $\{pq^* \mid p, q \text{ are paths in } \Gamma\}$ . (Recall that the elements of  $V$  are viewed as paths of length 0, so that this set includes elements of the form  $v$  with  $v \in V$ .)

If  $\Gamma$  is a finite graph, then  $L_K(\Gamma)$  is unital with  $\sum_{v \in V} v = 1_{L_K(\Gamma)}$ ; otherwise,  $L_K(\Gamma)$  is a ring with a set of local units consisting of sums of distinct vertices of the graph.

Many well-known algebras can be realized as the LPAs of a graph. The most basic graph configuration is shown below (the isomorphism for can be found in [1]).

**Example 1.** *The ring of Laurent polynomials  $K[x, x^{-1}]$  is the LPA of the graph given by a single loop graph.*

We will now outline some easily derivable basic facts about the endomorphism ring  $S$  of  $L := L_K(\Gamma)$ . Let  $\Gamma$  be any graph and  $K$  be any field. Denote by  $S$  the unital ring  $End(L_L)$ . Then we may identify  $L$  with subring of  $S$ , concretely, the following is a monomorphism of rings:

$$\begin{aligned} \phi : L &\rightarrow End(L_L) \\ x &\mapsto \lambda_x \end{aligned}$$

where  $\lambda_x : L \rightarrow L$  is the left multiplication by  $x$ , i.e., for every  $y \in L$ ,  $\lambda_x(y) = xy$  which is a homomorphism of right  $L$ -module. The map  $\phi$  is also a monomorphism because given a nonzero  $x \in L$  there exists an idempotent  $u \in L$  such that  $xu = x$ , hence  $0 \neq x = \lambda_x(u)$ .

### 3. RESULTS

According to Abrams and Rangaswamy [3]:

- A (possibly nonunital) ring  $R$  is called a *ring with local units* if, for each finite subset  $S$  of  $R$ , there exists an idempotent  $e$  of  $R$  such that  $S \subseteq eRe$ ;
- If  $R$  is a ring with local units then  $R$  is called *locally unit regular* if for each  $a \in R$  there is an idempotent (a local unit)  $v$  and local inverses  $u, u'$  such that  $uu' = v = u'u$ ,  $va = a = av$  and  $aua = a$  (see [3, Definition 6]).

**Theorem 1.** *Let  $\Gamma$  be an arbitrary graph,  $K$  be any field and  $S$  be the endomorphism ring of  $L := L_K(\Gamma)$ .*

- (1) *If  $S$  is locally unit regular, then  $L$  is locally unit regular. Moreover  $L$  is regular.*
- (2) *If  $L$  is locally unit regular, then  $vLv$  is locally unit regular for every non zero idempotent  $v$  of  $L$ .*

*Proof.* (1) Take  $x \in L$ . Since  $S$  is local unit regular, there exists an idempotent  $e \in S$  such that  $\lambda_x \in eSe$  and elements  $f, g \in eSe$  such that  $fg = e = gf$  and  $\lambda_x f \lambda_x = \lambda_x$ . Choose an idempotent  $u \in L$  such that  $x \lambda_{e(u)} = x = \lambda_{e(u)} x$  so  $x \in \lambda_{e(u)} L \lambda_{e(u)}$ . Note that,

$$\lambda_{f(u)} \lambda_{g(u)} = \lambda_{e(u)} = \lambda_{g(u)} \lambda_{f(u)}$$

and

$$\lambda_x = \lambda_x \lambda_{f(x)} = \lambda_x \lambda_{f(ux)} = \lambda_x \lambda_{f(u)} \lambda_x.$$

Since  $f \in eSe$ , there exists  $h \in S$  such that  $f = che$ . Then  $f(u) = e(u)h(u)e(u)$ , so

$$\lambda_{f(u)} = \lambda_{e(u)h(u)e(u)} = \lambda_{e(u)}\lambda_{h(u)}\lambda_{e(u)}$$

and we get  $\lambda_{f(u)} \in \lambda_{e(u)}L\lambda_{e(u)}$ . Similarly  $\lambda_{g(u)} \in \lambda_{e(u)}L\lambda_{e(u)}$ . Hence  $L$  is locally unit regular.

(2) Take any  $a \in vLv$ . Since  $L$  is locally unit regular, there exist an idempotent  $e$  and local inverses  $u, u'$  such that  $ea = a = ae$ ,  $uu' = e = u'u$  and  $aua = a$ . As  $ea = a$  and  $av = a$  which imply  $vea = va = a = ea$  and  $eav = ea$  respectively, we get  $ea = eav = vea$ . Now  $ea \in vLv$ , which implies  $e \in vLv$ . Then  $ve = e = ev$ . Let  $e^* = vev$ ,  $h = vuv$  and  $h' = veu'ev$ . Note that

$$e^*e^* = (vev)(vev) = vevev = veevv = vev = e^* \in vLv$$

$$hh' = (vuv)(veu'ev) = vuvveu'ev = vveu'ev = veu'^*$$

$$h'h = (veu'ev)(vuv) = veu'evuv = veu'euw = veu'^*$$

$$aha = a(vuv)a = vauav = vav = a,$$

which imply  $vLv$  is locally unit regular.  $\square$

**Definition 3.** A ring  $R$  is dependent if, for each  $a, b \in R$ , there are  $s, t \in R$ , not both zero, such that  $sa + tb = 0$ .

Let  $\Gamma$  be an arbitrary graph,  $K$  be any field and  $S$  be the endomorphism ring of  $L := L_K(\Gamma)$  considered as a right  $L$ -module. If  $S$  is dependent so is  $L$ . In fact, suppose  $S$  is dependent and  $a, b \in L$ . Then there are elements  $f, g \in S$ , not both zero, such that  $f\lambda_a + g\lambda_b = 0$ . If  $u_1$  and  $u_2$  are local units in  $L$  satisfying  $u_1a = a = au_1$  and  $u_2b = b = bu_2$ , then

$$f\lambda_a = f\lambda_{u_1a} = f\lambda_{u_1}\lambda_a = \lambda_{f(u_1)}\lambda_a$$

and

$$g\lambda_b = g\lambda_{u_2b} = g\lambda_{u_2}\lambda_b = \lambda_{g(u_2)}\lambda_b.$$

Now

$$\begin{aligned} 0 &= f\lambda_a + g\lambda_b \\ &= \lambda_{f(u_1)}\lambda_a + \lambda_{g(u_2)}\lambda_b, \end{aligned}$$

and hence  $L$  is dependent.

In the literature on von-Neumann regular rings, various conditions have been shown to characterize the subclass of unit regular rings. In [8, Theorem 6], Ehrlich showed that every unit regular ring  $R$  is dependent. In [10, Corollary 10], Henriksen shows that not all dependent regular rings are unit regular. The following observation gives one more such condition for dependent rings.

**Theorem 2.** If  $L_K(\Gamma)$  is locally unit regular, then it is dependent.

*Proof.* Let  $L_K(\Gamma)$  be locally unit regular and let some elements provide locally unit regular condition in the definition. Take  $a, b \in L_K(\Gamma)$ . If both  $a$  and  $b$  have local inverses in  $L_K(\Gamma)$ , then there exist  $u_1$  and  $u_2$  in  $L_K(\Gamma)$  such that  $u_1a = v$  and  $u_2b = v$  for local unit  $v$  in  $L_K(\Gamma)$ . So, we get  $sa + tb = 0$ , where  $s = u_1$  and  $t = -u_2$ . If one of the elements, say  $a$ , has no local inverse in  $L_K(\Gamma)$ , by definition of locally unit regularity, then we can write  $aua = a \Rightarrow aua = va \Rightarrow (au - v)a = 0$ . Now we get  $au - v \neq 0$ . Assume  $au - v = 0$ . So  $au = v$ , it is a contradiction. Then, for  $s = (au - v) \neq 0$  and  $t = 0$ , which implies  $sa + tb = 0$ .  $\square$

**Definition 4.** Let  $R$  be a ring with local units. We call  $R$  left (right) locally unit regular ring if for each  $a \in R$  there exist an idempotent  $v \in R$  and left (right) local inverses  $u, u'$  such that  $u'u = v$  ( $uu' = v$ ),  $va = a$  ( $av = a$ ) and  $aua = a$ .

**Definition 5.** ([12]) Let  $M$  be a right  $R$ -module, and let  $S = \text{End}_R(M)$ . Then  $M$  is called is a  $d$ -Rickart (or dual Rickart) module if the image in  $M$  of any single element of  $S$  is a direct summand of  $M$ . Clearly,  $R_R$  a  $d$ -Rickart module iff  $R$  is a regular ring.

**Definition 6.** Given paths  $p, q \in \Gamma$ , we say that  $q$  is an initial segment of  $p$  if  $p = qm$  for some path  $m \in \Gamma$ . It is well known that, given non-zero paths  $pq^*$  and  $mn^*$  in  $L_K(\Gamma)$ ,  $q$  is an initial segment of  $m$  if and only if  $(pq^*)(mn^*) \neq 0$ .

**Theorem 3.** Let  $\Gamma$  be a graph,  $K$  be any field and  $S$  be the endomorphism ring of  $L := L_K(\Gamma)$  considered as a right  $L$ -module. The following conditions are equivalent.

- (1)  $S$  is left locally unit regular.
- (2)  $S$  is regular and, for all paths  $x, y \in L$ ,  $Sx = Sy$  implies  $x$  is an initial segment of  $y$ .
- (3)  $L$  is dual-Rickart and, for all paths  $x, y \in L$ ,  $Sx = Sy$  implies  $x$  is an initial segment of  $y$ .

*Proof.* (1)  $\Rightarrow$  (2) Assume that  $S$  is left locally unit regular. Hence  $S$  is regular and  $L$  is left locally unit regular by Theorem 1. Let  $x, y \in L$  be two paths. Then there exist an idempotent  $v \in L$  and left local inverses  $v_1, v_2 \in L$  such that  $vy = y$ ,  $v_2v_1 = v$  and  $y = yv_1y$ . If  $Sx = Sy$ , then  $x = f(y)$  for some  $f \in S$ . Now  $y = yv_1y$  implies  $f(y) = f(yv_1y)$ , and so  $x = \underbrace{f(yv_1)}_{\in L}y$ . Hence  $x$  is an initial segment of  $y$ .

(2)  $\Rightarrow$  (3) This follows from [17, Corollary 3.2].

(3)  $\Rightarrow$  (1) Assume that  $L$  is dual-Rickart. Then  $f(L)$  is a direct summand of  $L$ , where  $f \in S$ . Let  $e$  be an idempotent in  $S$  with  $f(L) = eL$ . Let  $x \in L$ . Then there exists  $y \in L$  such that  $f(x) = e(y)$ . Now

$$(ef)(x) = e(f(x)) = e(e(y)) = e(y) = f(x),$$

which implies  $ef = f$ . Let  $h$  be the left inverse of  $f$  and  $g = fe$ . Then  $gh = e$  and  $fhf = f$ .  $\square$

**Definition 7.** ([13]) An endomorphism  $\alpha$  of a module  $M$  is called *morphic* if  $M/M\alpha \cong \text{Ker}(\alpha)$ , equivalently there exists  $\beta \in \text{End}(M)$  such that  $M\beta = \text{Ker}(\alpha)$  and  $\text{Ker}(\beta) = M\alpha$  by [13, Lemma 1]. The module  $M$  is called a *morphic module* if every endomorphism is morphic. If  $R$  is a ring, an element  $a$  in  $R$  is called *left morphic* if right multiplication  $\cdot a : {}_R R \rightarrow {}_R R$  is a morphic endomorphism, that is if  $R/Ra \cong l(a)$ . The ring itself is called a *left morphic ring* if every element is left morphic, that is if  ${}_R R$  is a morphic module.

Note that if  $S$  is dependent then  $L_K(\Gamma)$  is morphic by [14, Corollary 3.5].

**Theorem 4.** Let  $\Gamma$  be any graph and let  $K$  be any field. If  $L_K(\Gamma)$  is left morphic and regular ring then  $L_K(\Gamma)$  is left locally unit regular ring.

*Proof.* Let  $L_K(\Gamma) = L$  be left morphic and regular ring. Then each  $a \in L$  is both regular and morphic. So, there exist an  $x \in L$  such that  $a = axa$  and for some  $b \in L$ ,  $La = \text{ann}(b)$  and  $Lb = \text{ann}(a)$ . Let  $u = xax + b$ . Then  $a = au$ . To see that  $u$  is left local inverse, since  $L$  has local units, choose an idempotent  $v \in L$  such that  $va = a$ . Then we get,  $0 = va - a = va - axa = (v - ax)a$ , so  $v - ax \in \text{ann}(a) = Lb$  and there exists an element  $y \in L$  such that  $v - ax = yb$ . We take  $u' = a + y(v - ax)$ . We show that  $u'u = v$ :

$$\begin{aligned} u'u &= (a + y(v - ax))(xax + b) \\ &= axax + ab + y(v - ax)xax + y(v - ax)b \\ &= ax + ab + yvxax - yxaxax + yvb - yxab \\ &= ax + yb = v \end{aligned}$$

Hence  $L = L_K(\Gamma)$  is left locally regular ring.  $\square$

**Theorem 5.** Let  $\Gamma$  be a graph,  $K$  be any field and  $S$  be the endomorphism ring of  $L := L_K(\Gamma)$  considered as a right  $L$ -module. If  $L_K(\Gamma)$  is morphic and image projective then  $S$  is left morphic.

*Proof.* Let  $L := L_K(\Gamma)$  be morphic and image projective. Given any  $\alpha \in S$ , since  $L$  is morphic, we may choose an  $\beta \in S$  such that,  $L\alpha = \text{ker}(\beta)$  and  $L\beta = \text{ker}(\alpha)$ . Since  $\alpha\beta = 0$ ,  $S\alpha \subset \text{ann}^S(\beta)$ . Conversely, if  $\gamma \in \text{ann}^S(\beta)$  then  $\gamma\beta = 0$  so  $L\gamma \subset \text{ker}(\beta) = L\alpha$  and hence  $\gamma \in S\alpha$  because  $L$  is image projective. Thus  $S\alpha = \text{ann}^S(\beta)$ . We may see  $S\beta = \text{ann}^S(\alpha)$  in the same way. Hence  $S$  is left morphic.  $\square$

**Definition 8.** ([16, Definition 4.1]) If  $R$  is a ring with local units then  $R$  is called *directly finite* if for each  $x, y \in R$  there is an idempotent  $u$  such that  $xu = x = ux$  and  $yu = y = uy$ , we have that  $xy = u$  implies  $yx = u$ .

**Theorem 6.** Let  $\Gamma$  be a graph,  $K$  be any field and  $S$  be the endomorphism ring of  $L := L_K(\Gamma)$  considered as a right  $L$ -module. If  $S$  is a directly finite ring then  $L_K(\Gamma)$  is directly finite.

*Proof.* Take any  $x, y$  in  $L_K(\Gamma)$ . Since  $S$  is a direct finite ring, there is an idempotent  $\varepsilon$  in  $S$  such that  $\lambda_x \varepsilon = \lambda_x = \varepsilon \lambda_x$  and  $\lambda_y \varepsilon = \lambda_y = \varepsilon \lambda_y$ , we have that  $\lambda_x \lambda_y = \varepsilon$  implies  $\lambda_y \lambda_x = \varepsilon$ . For an idempotent  $u$  with  $xu = x = ux$  and  $yu = y = uy$ ,

$$\begin{aligned} \lambda_x \lambda_y = \varepsilon &\Rightarrow \lambda_x \lambda_y \lambda_x = \varepsilon \lambda_x \Rightarrow \lambda_x = \varepsilon \lambda_{uv} \Rightarrow \lambda_x = \lambda_{\varepsilon(u)} \lambda_x \\ \lambda_x &= \varepsilon \lambda_x = \lambda_{\varepsilon(x)} = \lambda_{\varepsilon(xu)} = \lambda_{\varepsilon(x)} \lambda_{\varepsilon(u)} = \varepsilon \lambda_x \lambda_{\varepsilon(u)} \end{aligned}$$

So,  $\lambda_x \lambda_{\varepsilon(u)} = \lambda_x = \lambda_{\varepsilon(u)} \lambda_x$ . Similarly  $\lambda_y \lambda_{\varepsilon(u)} = \lambda_y = \lambda_{\varepsilon(u)} \lambda_y$ . Assume that,  $\lambda_x \lambda_y = \lambda_{\varepsilon(u)}$ . We then see that  $\lambda_y \lambda_x = \lambda_{\varepsilon(u)}$ .

$$\lambda_y \lambda_x = \lambda_y \lambda_{\varepsilon(u)} \lambda_x = \lambda_y \lambda_x \lambda_{\varepsilon(u)} = \varepsilon \lambda_{\varepsilon(u)} = \lambda_{\varepsilon^2(u)} = \lambda_{\varepsilon(u)},$$

as desired. □

One hopes that if  $L_K(\Gamma)$  is directly finite then  $L_K(\Gamma)$  is locally unit regular but this is not true. Because  $K[x, x^{-1}]$  is a commutative Leavitt path algebra (of the graph with one vertex and one loop) clearly directly finite. But it is not von Neumann regular ring.

**Corollary 1.** *Let  $\Gamma$  be a graph,  $K$  be any field and  $S$  be the endomorphism ring of  $L := L_K(\Gamma)$  considered as a right  $L$ -module. If  $S$  is a directly finite ring, then  $\Gamma$  is no exit.*

*Proof.* Let  $S$  be a directly finite ring. Then  $L_K(\Gamma)$  is a directly finite ring. So, by [16, Proposition 4.3],  $\Gamma$  is no exit. □

**Definition 9.**  *$R$  is said to be a (left) exchange ring if for any direct decomposition  $A = M \oplus N = \bigoplus_{i \in I} A_i$  of any left  $R$ -module  $A$ , where  $R \cong M$  as left  $R$ -modules and  $I$  is a finite set, there always exist submodules  $B_i$  of  $A_i$  such that  $A = M \oplus (\bigoplus_{i \in I} B_i)$ .*

**Theorem 7.** *Let  $\Gamma$  be an infinite graph,  $K$  be any field and  $S$  be the endomorphism ring of  $L := L_K(\Gamma)$  considered as a right  $L$ -module. Then*

- (1) *If  $S$  is an exchange ring then  $L$  is directly finite.*
- (2) *If  $L$  is a direct finite ring then  $L$  is an exchange ring.*

*Proof.* (1) Let  $S$  be an exchange ring. Then, by [5, Proposition 2.10],  $L_K(\Gamma)$  is an exchange ring. For every  $x, y \in L$  and an idempotent  $u \in L$  such that  $xu = x = ux$  and  $yu = y = uy$  we have that  $xy = u$ . We show that  $yx = u$ . Since  $L$  is an exchange ring, there exist  $r, s \in L$  such that  $u = rx = s + x - sx$ . So,  $u = rx \Rightarrow uy = rxy \Rightarrow y = ru \Rightarrow yx = rux = rx = u$ , as desired.

(2) Let  $L$  be a direct finite ring. For any  $x, y \in L$  and an idempotent  $u \in L$  such that  $xu = x = ux$  and  $yu = y = uy$  we have that  $xy = u$  implies  $yx = u$ . We show that  $L$  is an exchange ring. For any  $x \in L$  taking  $r = y$  and  $s = u$ , we get  $u = rx = s + x - sx$ . So,  $L$  is an exchange ring. □

**Corollary 2.** *Let  $\Gamma$  be infinite graph,  $K$  be any field and  $S$  be the endomorphism ring of  $L := L_K(\Gamma)$  considered as a right  $L$ -module. Then the following conditions are equivalent.*

- (1)  $S$  is an exchange ring.
- (2)  $L_K(\Gamma)$  is an exchange ring.
- (3)  $L_K(\Gamma)$  is a directly finite ring.
- (4)  $E$  is no exit

*Proof.* (1)  $\Leftrightarrow$  (2) This is [5, Proposition 2.10].

(2)  $\Leftrightarrow$  (3) This follows from Theorem 7 (1) and Theorem 7 (2).

(3)  $\Leftrightarrow$  (4) This is [16, Teorem 4.12]. □

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