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(WEAKLY) $n-$NIL CLEANNESS OF THE RING $\mathbb{Z}_m$

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Abstract. Let R be an associative ring with identity. For a positive integer $n \geqslant 2$, an element $a \in R$ is called $n-$potent if $a^n = a$. We define R to be (weakly) $n-$nil clean if every element in R can be written as a sum (a sum or a difference) of a nilpotent and an $n-$potent element in R. This concept is actually a generalization of weakly nil clean rings introduced by Danchev and McGovern, [6]. In this paper, we completely determine all $n, m \in \mathbb{N}$ such that the ring of integers modulo m, $\mathbb{Z}_m$ is (weakly) $n-$nil clean.

1. Introduction

Let R be an associative ring with identity. Throughout this text, the notations $U(R)$, $J(R)$, $Id(R)$ and $N(R)$ will stand for the set of units, the Jacobson radical, the set of idempotents and the set of nilpotents of R, respectively. Following [14], we define an element r of a ring R to be clean if there is an idempotent $e \in R$ and a unit $u \in R$ such that $r = u + e$. A clean ring is defined to be one in which every element is clean. Similarly, an element r in a ring R is said to be nil clean if $r = e + b$ for some idempotent $e \in R$ and a nilpotent element $b \in R$. A ring R is nil clean if each element of R is nil clean. In [2], Breaz, Danchev and Zhou defined a ring R to be weakly nil clean if each element $r \in R$ can be written as $r = b + e$ or $r = b - e$ for $b \in N(R)$ and $e \in Id(R)$. We refer the reader to [8, 1, 3, 5, 7, 4, 2] for a survey on nil clean and weakly nil clean rings.

For $a \in R$ and a positive integer $n \geqslant 2$, we say that a is $n-$potent if $a^n = a$. Moreover, a is called (weakly) $n-$nil clean if it is a sum (a sum or a difference) of $n-$potent element and a nilpotent element in R. We define R to be (weakly) $n-$nil clean if every element in R is (weakly) $n-$nil clean. Weakly $n-$clean rings are defined in a similar way. Obviously, the (weakly) $2-$nil clean rings are the same as the (weakly) nil clean rings. R is called a generalized nil clean if every element in R is $n-$nil clean for some $n \in \mathbb{N}$. The class of $n-$nil clean and generalized nil clean rings were firstly defined and studied in [9] by Hirano, Tominaga and Yaqub

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in 1988. Some Other authors called generalized nil clean rings as weak periodic rings. A ring R is called periodic if for every $x \in R$, there are distinct integers m and k such that $x^m = x^k$. It is proved that a periodic ring is weak periodic and that the converse is true if in any expansion $r = b + s$ for potent s and $b \in N(R)$, we have $bs = sb$.

In this paper, we focus our attention on the ring $\mathbb{Z}_m$ of integers modulo a positive integer m. We use the well Known Hensel's Lemma to completely determine when the ring $\mathbb{Z}_m$ is (weakly) $n-$nil clean ring for any $m, n \in \mathbb{N}$. Moreover, we determine all $m, n \in \mathbb{N}$ such that every element $r \in \mathbb{Z}_m$ is of the form $r = b \pm s$ where $b \in N(R)$ and $s^n = -s$. Next, we apply our results for some special values of m and n.

In the next section, we study weakly $n-$nil clean rings and introduce some fundamental facts and examples concerning this class of rings. Among many other properties, we determine some conditions on n, R and G under which the group ring RG is (weakly) $n-$nil clean.

2. Weakly $n-$Nil Clean Rings

In this section, we study some of the basic properties of weakly $n-$nil clean rings. Moreover, we give some necessarily examples.

Definition 1. *Let R be a ring and $n \in \mathbb{N}$ where $n \geqslant 2$. An element $r \in R$ is called weakly $n-$nil clean if there exist $b \in N(R)$ and an $n-$potent element s of R such that $r = b + s$ or $r = b - s$. A ring R is called weakly $n-$nil clean if all of its elements are weakly $n-$nil clean.*

For $n \geqslant 2$, let s be an $n-$potent and b be a nilpotent. For $r \in R$, we write $r = b \pm s$ if r is either a sum $b + s$ or a difference $b - s$. Obviously, every $n-$nil clean ring is weakly $n-$nil clean. Since the ring $\mathbb{Z}_6$ is a weakly nil clean ring that is not nil clean, then trivially $\mathbb{Z}_6$ is a weakly $2-$nil clean ring which is not $2-$nil clean. For a non trivial example, one can easily verify that the ring $\mathbb{Z}_3$ is weakly $4-$nil clean but not $4-$nil clean. Moreover, if a ring R is a weakly n-nil clean, then it is weakly n-clean. Indeed, if we let $x \in R$, then $x - 1 = b \pm s$ where $b \in N(R)$ and $s^n = s$. So, $x = (b + 1) \pm s$ where $b + 1 \in U(R)$. The converse is not true in general. For example, simple computations show that the ring $R = T_2(\mathbb{Z}_3)$ is weakly 5-clean which is not weakly 5-nil clean.

Next, we give some properties of the class of weakly $n-$nil clean rings. The proof of the following proposition is straightforward.

Proposition 1. *Let R and S be two rings, $\mu : R \to S$ be a ring epimorphism and $n \geqslant 2$. If R is weakly $n-$nil clean, then S is weakly $n-$nil clean.*

The following Properties (2), (3) and (4) in Corollary 1 are direct consequences of Proposition 1. The proofs of Properties (1) and (5) are similar to that of (weakly) $g(x)$-nil clean appeared in [10, 11, 12].

Corollary 1. *Let R and S be ring and let $n \geqslant 2$. The following hold:*

(1) If I is an ideal in R and R is weakly $n-$nil clean, then R/I is weakly $n-$nil clean. Moreover, the converse holds if I is nil and potent elements lift modulo I.

(2) If the upper triangular matrix ring $T_n(R)$ is weakly $n-$nil clean, then R is weakly $n-$nil clean.

(3) If the skew formal power series $R[[x,\alpha]]$ (or in particular $R[[x]]$) over R is weakly $n-$nil clean, then R is weakly $n-$nil clean.

(4) Let M be an (R,S)-bimodule and $T = \begin{bmatrix} A & M \\ 0 & B \end{bmatrix}$ be the formal triangular matrix ring. If T is weakly $n-$nil clean, then R and S are weakly $n-$nil clean.

(5) If R is commutative and M an R-module. Then the idealization $R(M)$ of R and M is weakly $n-$nil clean if and only if R is weakly $n-$nil clean.

Proposition 2. *Let $R = \prod_{i\in I} R_i$ be a direct product of rings with I is finite and $|I| \geq 2$ and let $n \geqslant 2$. R is weakly $n-$nil clean if and only if there exist $k \in I$ such that R_k is weakly $n-$nil clean and R_j is $n-$nil clean for $j \neq k$.*

Proof. $\Rightarrow)$: For each $i \in I$, R_i is a homomorphic image of $\prod_{i\in I} R_i$ under the projection homomorphism. Hence, R_i is weakly $n-$nil clean by Proposition 1. Without loss of generality, assume that neither R_1 nor R_2 are $n-$nil clean. Then there exist $r_1 \in R_1$ and $r_2 \in R_2$ such that r_1 is not a sum of a nilpotent and an $n-$potent and r_2 is not a difference of a nilpotent and an $n-$potent. Thus (r_1, r_2) is not weakly $n-$nil clean in $R_1 \times R_2$, a contradiction.

$\Leftarrow)$: Assume that R_k is weakly $n-$nil clean for a fixed index $k \in I$. Thus R_j is $n-$nil clean for all $j \neq k$. Let $r = (r_i) \in R$. Then there exist $b_k \in N(R_k)$ and an $n-$potent s_k such that $r_k = b_k + s_k$ or $r_k = b_k - s_k$. If $r_k = b_k + s_k$, then for each $i \in I - \{k\}$, write $r_i = b_i + s_i$ where $b_i \in N(R_i)$ and $s_i^n = s_i$. Therefore, $r = (b_i) + (s_i)$ is a sum of a nilpotent and an $n-$potent. If $r_k = b_k - s_k$, then for each $i \in I - \{k\}$, write $r_i = b_i - s_i$ where $b_i \in N(R_i)$ and $s_i^n = s_i$. Consequently, $r = (b_i) - (s_i)$ is a difference of a nilpotent and an $n-$potent. Therefore, R is weakly $n-$nil clean. □

Definition 2. *Let R be a ring and let $m \in \mathbb{N}$. Then R is said to have the nil m-involution property if for every $r \in R$, we have $r = u + v$ where $u \in 1 \pm N(R)$ and $v^m = 1$.*

We now justify the relation between weakly $n-$nil clean rings and rings with nil $(n-1)$-involution property for an odd $n \in \mathbb{N}$.

Proposition 3. *Let R be a ring and let n be an odd integer with $n \geq 2$. If R has the nil $(n-1)$-involution property, then R is (weakly) $n-$nil clean. If moreover, aR (or Ra) contains no non trivial idempotent for every non unit $a \in R$, then the two statements are equivalent.*

Proof. Suppose R has the nil $(n-1)$-involution property and let $r \in R$. Write $r + 1 = u + v$ where $u \in 1 \pm N(R)$ and $v^{n-1} = 1$. Then $r = (u-1) + v$ where $u - 1 \in N(R)$ and v is clearly an $n-$potent element in R.

Now, we assume that for every non unit $a \in R$, aR (or Ra) contains no non trivial idempotents and suppose R is weakly $n-$nil clean. Let $a \in R$ and write $a-1 = b \pm s$ where $b \in N(R)$ and $s^n = s$. Then $as^{n-1} = (b+1)s^{n-1} \pm s$ and so $a(1-s^{n-1}) = (b+1)(1-s^{n-1}) = u(1-s^{n-1})$ where $u \in U(R)$. Since clearly $u(1-s^{n-1})u^{-1} = a(1-s^{n-1})u^{-1} \in aR$ is an idempotent, then by assumption $u(1-s^{n-1})u^{-1} = 0$ or $u(1-s^{n-1})u^{-1} = 1$. Therefore $s^{n-1} = 1$ or $s^{n-1} = 0$. In the last case, we get $s = s^n = 0$ and so $a = b+1$ is a unit, a contradiction. Thus, $a = (b+1) + (\pm s)$ where $(\pm s)^{n-1} = 1$ since $n-1$ is even. The case when Ra contains no non trivial idempotent for every non unit $a \in R$ is similar. Therefore, R has the nil $(n-1)$-involution property. □

It is easy to see that the ring $\mathbb{Z}_4$ is a (weakly) $4-$nil clean ring with $a\mathbb{Z}_4$ contains no non trivial idempotent for every $a \in \mathbb{Z}_4$. But, $\mathbb{Z}_4$ does not have the nil 3-involution property. Therefore, the equivalence in Proposition 3 need not be hold for an even integer n.

Let R be a ring and G be a finite cyclic group. In the following Proposition, we determine conditions under which the group ring RG is (weakly) $n-$nil clean. We recall that R is called an $n-$potent ring if $a^n = a$ for every $a \in R$.

Proposition 4. *Let G any cyclic group of order p (prime).*

(1) If R is a Boolean ring, then RG is a $2^{p-1}-$potent ring (and so is (weakly) $2^{p-1}-$nil clean).

(2) If R is a commutative $3-$potent ring of characteristic 3, and $p \neq 3$, then RG is a $3^{p-1}-$potent ring (and so is (weakly) $3^{p-1}-$nil clean).

Proof. (1) See Proposition 3.17 in [10].

(2) Let $G = \left\{1, g, g^2, ..., g^{p-1}\right\}$ where $g^p = 1$ and let $x = a_0 + a_1 g + a_2 g^2 + ... + a_{p-1}g^{p-1} \in RG$. First, we prove by induction that $x^{3^k} = \sum\limits_{i=0}^{p-1} a_i g^{i*(3^k)}$ for all $k \in \mathbb{N}$. Let $k = 1$. Since R is $3-$potent ring of characteristic 3, one can see that $x^3 = a_0 + a_1 g^3 + a_2 g^6 + ... + a_{p-1} g^{3(p-1)} = \sum\limits_{i=0}^{p-1} a_i g^{3i}$. Suppose the result is true for k. Then $x^{3^{k+1}} = (x^3)^{3^k} = \sum\limits_{i=0}^{p-1} a_i (g^3)^{i*(3^k)} = \sum\limits_{i=0}^{p-1} a_i g^{i*(3^{k+1})}$. By Fermat Theorem, $3^{p-1} = 1 + np$ for some integer n. Thus, $x^{3^{p-1}} = \sum\limits_{i=0}^{p-1} a_i g^{i*(3^{p-1})} = \sum\limits_{i=0}^{p-1} a_i g^{i*(1+np)} = \sum\limits_{i=0}^{p-1} a_i g^i = x$. Therefore, RG is a $3^{p-1}-$potent ring. □

By Proposition 4, we conclude that the ring $\mathbb{Z}_2(C_3)$ is (weakly) $4-$nil clean and $\mathbb{Z}_3(C_2)$ is (weakly) $3-$nil clean.

Proposition 5. *Let R be a ring and let $n \geqslant 2$. If R is (weakly) $n-$nil clean, then $J(R)$ is nil.*

Proof. Let $a \in J(R)$. Then $a = b \pm s$ where $b \in N(R)$ and $s^n = s$. If $a = b - s$, then $a + s \in N(R)$. If we choose $m \in \mathbb{N}$ such that $(a+s)^m = 0$, then clearly $s^m \in J(R)$. If $m \lneqq n$, then $s^{n-1} \in J(R)$. Since also $s(1 - s^{n-1}) = 0$ and $1 - s^{n-1} \in U(R)$, then $s = 0$. If $m \geq n$, then we can similarly see that $s = 0$. Hence $a = b \in N(R)$. Similarly, the case $a = b + s$ gives $a \in N(R)$ and so $J(R)$ is nil. □

3. When the ring $\mathbb{Z}_m$ is (weakly) $n-$nil clean

In the main Theorem of this section, we completely determine all $n, m \in \mathbb{N}$ such that the ring $\mathbb{Z}_m$ is (weakly) n-nil clean. We recall that for $m \in \mathbb{N}$, the set of all positive integers less than or equal m that are relatively prime to m is a group under multiplication modulo m. it is denoted by $\mathbb{Z}_m^{\times}$ and is called the group of units modulo m. This group is cyclic if and only if m is equal to 2, 4, p^k, or $2p^k$ where p^k is a power of an odd prime. A generator of this cyclic group is called a primitive root modulo m. The order of $\mathbb{Z}_m^{\times}$ is given by Euler's totient function $\varphi(m)$. It is easy to see that for any prime integer p and any $k \in \mathbb{N}$, $\varphi(p^k) = p^{k-1}(p-1)$. For more details one can see [13].

Lemma 1. *For any $n, k \in \mathbb{N}$, the ring $\mathbb{Z}_{2^k}$ is $n-$nil clean.*

Proof. For any $n \in \mathbb{N}$, at least 0 and 1 are $n-$potent elements in $\mathbb{Z}_{2^k}$. Since $N(\mathbb{Z}_{2^k}) = \{0, 2, 4, ..., 2(2^{k-1} - 1)\}$, then clearly any element in $\mathbb{Z}_{2^k}$ is a sum of a nilpotent and an $n-$potent. □

The following lemma is a special case of the well known Hensel's Lemma.

Lemma 2. *Let $n, k \in \mathbb{N}$ and p be an odd prime integer. Consider the congruence $f(x) \equiv 0 (\mathrm{mod}\, p)$ where $f(x) \in \mathbb{Z}[x]$. If r is a solution of the congruence with $f'(r)$ is not congruent to $0(\mathrm{mod}\, p)$, then there exists a unique integer s such that $f(s) \equiv 0(\mathrm{mod}\, p^k)$ and $r \equiv s(\mathrm{mod}\, p)$.*

In particular, for a prime integer p and $1 \leq m \leq p-1$, let r be a solution of $x^m - 1 \equiv 0(\mathrm{mod}\, p)$. Then mr^{m-1} is not congruent to $0(\mathrm{mod}\, p)$. Hence, r corresponds to a unique solution s of $x^m - 1 \equiv 0(\mathrm{mod}\, p^k)$ such that $r \equiv s(\mathrm{mod}\, p)$.

The following Lemma is well known in number theory. However, we give the proof for the sake of completeness.

Lemma 3. *Let $n, k \in \mathbb{N}$ and p be any prime integer and let $d = \gcd(n, p^{k-1}(p-1))$. Then*

(1) The polynomial $x^n - 1 \in \mathbb{Z}_{p^k}[x]$ has d solutions in $\mathbb{Z}_{p^k}$.

(2) If $\frac{p^{k-1}(p-1)}{d}$ is even, then the polynomial $x^n + 1 \in \mathbb{Z}_{p^k}[x]$ has d solutions in $\mathbb{Z}_{p^k}$. Otherwise, it has no solutions.

Proof. (1) Consider the cyclic group of units $\mathbb{Z}_{p^k}^{\times}$ with order $\varphi(p^k) = p^{k-1}(p-1)$. Let g be a generator for $\mathbb{Z}_{p^k}^{\times}$ and let $a = g^m \in \mathbb{Z}_{p^k}^{\times}$ be a solution of $x^n \equiv 1(\mathrm{mod}\, p^k)$. Then $a^n = g^{mn} \equiv 1(\mathrm{mod}\, p^k)$ and so $p^{k-1}(p-1)$ divides mn. If we let $d =$

$\gcd(n, p^{k-1}(p-1))$, then $\frac{p^{k-1}(p-1)}{d}$ divides m. Therefore, the solution set of $x^n - 1$ in $\mathbb{Z}_{p^k}$ forms a subgroup generated by $g^{\frac{p^{k-1}(p-1)}{d}}$. The result follows since this subgroup is clearly of order d.

(2) Consider again the generator g of the cyclic group of units $\mathbb{Z}_{p^k}^{\times}$. Since $g^{p^{k-1}(p-1)} \equiv 1 (\bmod p^k)$, then g must satisfy $g^{\frac{p^{k-1}(p-1)}{2}} \equiv -1 (\bmod p^k)$. Hence, $x = g^m$ is a solution of $x^n \equiv -1 (\bmod p^k)$ if and only if $p^{k-1}(p-1)$ divides $2mn$ and so $\frac{p^{k-1}(p-1)}{d}$ must divides $2m$. If $\frac{p^{k-1}(p-1)}{d}$ is not even, then $x^n \equiv -1 (\bmod p^k)$ has no solutions. However, if $\frac{p^{k-1}(p-1)}{d}$ is even, then $g^{\frac{p^{k-1}(p-1)}{2d}}$ is one solution of $x^n \equiv -1 (\bmod p^k)$. The other solutions can be obtained by multiplying by the d solutions of $x^n \equiv 1 (\bmod p^k)$. □

Theorem 1. *Let $n, k \in \mathbb{N}$ and p be any odd prime integer. If $d = \gcd(n-1, p^{k-1}(p-1))$, then $\mathbb{Z}_{p^k}$ is $n-$nil clean if and only if $d = p^t(p-1)$ for some $0 \leq t \leq k-1$.*

Proof. To be brief, let S denotes the set of all zeros of $x^n - x$ in $\mathbb{Z}_{p^k}$ and T denotes the set of sums of every element in $N(\mathbb{Z}_{p^k})$ to every element in S.

$\Leftarrow$) : Suppose $d = \gcd(n, p^{k-1}(p-1)) = p-1$. By Lemma 3, The multiplicative group G of roots of unity modulo p^k is of order $p-1$ and so $a^{p-1} \equiv 1 (\bmod p^k)$ for all $a \in G$. Now, By Fermat Theorem, any $a \in G$ is also a solution of $x^{p-1} \equiv 1 (\bmod p)$. By Lemma 2, the $p-1$ solutions of $x^{p-1} \equiv 1 (\bmod p)$ correspond uniquely to the $p-1$ solutions of $x^{p-1} \equiv 1 (\bmod p^k)$. Hence, the $p-1$ solutions of $x^{p-1} \equiv 1 (\bmod p^k)$ are congruent to $1, 2, ..., p-1$ in some order. Now, $N(\mathbb{Z}_{p^k}) = \{0, p, 2p, ..., (p^{k-1}-1)p\}$ is of order p^{k-1}. If $n_1 + a = n_2 + b$ for some $a, b \in S$ and $n_1, n_2 \in N(\mathbb{Z}_{p^k})$, then $a - b \equiv n_n - n_1 \equiv 0 (\bmod p)$. Thus, $a \equiv b (\bmod p)$ which is true only if $a = b = 0$. Therefore, T has exactly $pp^{k-1} = p^k$ distinct elements and $\mathbb{Z}_{p^k}$ is $n-$nil clean. Next, suppose $d = p^t(p-1)$ for some $1 \leq t \leq p^k$. If $a^{p^t(p-1)} \equiv 1 (\bmod p^k)$, then $a^{p-1} \equiv (a^{p-1})^{p^t} \equiv 1 (\bmod p)$. Again, by Lemma 2, the $p-1$ solutions corresponds uniquely to $p-1$ distinct solutions of $x^{p^t(p-1)} \equiv 1 (\bmod p^k)$. Hence, similar to the above argument, we conclude that $\mathbb{Z}_{p^k}$ is $n-$nil clean.

$\Rightarrow$) : Suppose $d = mp^t$ for some $m \mid (p-1)$ with $m \neq p-1$ and $0 \leq t \leq p^{k-1}$. If $t = 0$, then T has at most $(m+1)p^{k-1} \lneqq p^k$ elements and so $\mathbb{Z}_{p^k}$ is not $n-$nil clean. Let $t \gneqq 0$ and consider $x^{mp^t} \equiv 1 (\bmod p^k)$. Then $x^m \equiv (x^m)^{p^t} \equiv 1 (\bmod p)$ has at most m solutions $(\bmod p)$. By Lemma 2, any solution of $x^{mp^t} \equiv 1 (\bmod p^k)$ is congruent to one of the m solutions of $x^m \equiv 1 (\bmod p)$. Choose $1 \leq c \leq p-1$ such that c is not a solution of $x^m \equiv 1 (\bmod p)$ and suppose $c = a + f$ for some $a \in S$ and $f \in N(\mathbb{Z}_{p^k})$. If $a = 0$, then $c \in N(\mathbb{Z}_{p^k})$, a contradiction. Suppose $a \neq 0$ and let $1 \leq r \leq p-1$ such that $r^m \equiv 1 (\bmod p)$ and $a \equiv r (\bmod p)$. Then $c \equiv r + f \equiv r (\bmod p)$ which is a contradiction. Hence, again $\mathbb{Z}_{p^k}$ is not $n-$nil clean. □

Corollary 2. *For any even integer n and odd prime p, the ring $\mathbb{Z}_{p^k}$ is not $n-$nil clean.*

Definition 3. *Let R be a ring and $n \geqslant 2$. R is called (x^n+x)-nil clean if for every $r \in R$, $r = b + s$ where $b \in N(R)$ and $s^n = -s$.*

By direct computations one can easily verify that for any even integer n, R is (x^n+x)-nil clean if and only if R is $n-$nil clean. However for any odd integer n and odd prime integer p, we prove in the following lemma that $\mathbb{Z}_{p^k}$ is never (x^n+x)-nil clean.

Lemma 4. *For any $k \in \mathbb{N}$ and $n \geqslant 2$, the ring $\mathbb{Z}_{2^k}$ is (x^n+x)-nil clean if and only if $\gcd(n-1, 2^k) \neq 2^k$.*

Proof. The proof follows directly by (2) in Lemma 3. □

Theorem 2. *Let p be a prime integer and $k, n \in \mathbb{N}$ where n is odd. Then $\mathbb{Z}_{p^k}$ is never (x^n+x)-nil clean.*

Proof. $\Leftarrow$) : By lemma 3, $x^{n-1} \equiv -1 (\bmod\, p^k)$ has a solution if $\frac{p^{k-1}(p-1)}{d}$ is even where $d = \gcd(n-1, p^{k-1}(p-1))$. Hence, clearly if $d = p^t(p-1)$ for some $0 \leq t \leq k-1$, then $\mathbb{Z}_{p^k}$ is not (x^n+x)-nil clean. Suppose $d = mp^t$ for some $m \mid p-1$ with $m \neq p-1$ and $0 \leq t \leq k-1$. Then $x^m \equiv (x^m)^{p^t} \equiv -1 (\bmod\, p)$. Clearly, this congruence has less than $p-1$ solutions. Thus, as in the proof of the similar case in Theorem 1, we conclude that $\mathbb{Z}_{p^k}$ is not (x^n+x)-nil clean. □

Corollary 3. *Let $m, n, k \in \mathbb{N}$ and write $m = p_1^{r_1} p_2^{r_2} ... p_t^t$ where $p_1, p_2, ..., p_t$ are distinct prime integers. Then the ring $\mathbb{Z}_m$ is $n-$nil clean if and only if for all $i = 1, 2, ..., t$, $\gcd(n-1, p_i^{r_i-1}(p_i-1)) = p_i^l(p_i-1)$ for some $0 \leq l \leq r_i - 1$.*

Proof. We have $\mathbb{Z}_m \simeq \mathbb{Z}_{p_1^{r_1}} \times \mathbb{Z}_{p_2^{r_2}} \times ... \times \mathbb{Z}_{p_t^{r_t}}$. By Proposition (2.6) in [10], $\mathbb{Z}_m$ is $n-$nil clean if and only if $\mathbb{Z}_{p_i^{r_i}}$ is $n-$nil clean for all $i = 1, 2, ..., t$. Hence, the result follows by Theorem 1 and Lemma 1. □

As special cases, we have

Corollary 4. *Let $n \in \mathbb{N}$ and consider the ring $\mathbb{Z}_n$. Then*
(1) For any $m \in \mathbb{N}$, $\mathbb{Z}_n$ is $2m-$nil clean if and only if $n = 2^k$ for $k \in \mathbb{N} \cup \{0\}$.
(2) $\mathbb{Z}_n$ is $3-$nil clean if and only if $n = 2^k \times 3^m$ for $k, m \in \mathbb{N} \cup \{0\}$.
(3) $\mathbb{Z}_n$ is $5-$nil clean if and only if $n = 2^k \times 3^m \times 5^l$ for $k, m, l \in \mathbb{N} \cup \{0\}$.
(4) $\mathbb{Z}_n$ is $7-$nil clean if and only if $n = 2^k \times 3^m \times 7^l$ for $k, m, l \in \mathbb{N} \cup \{0\}$.

For $n, m \in \mathbb{N}$, we next clarify when the ring $\mathbb{Z}_m$ is weakly $n-$nil clean.

Theorem 3. *Let $n, k \in \mathbb{N}$, p be any odd prime integer and $d = \gcd(n-1, p^{k-1}(p-1))$. Then*

(1) $\mathbb{Z}_{p^k}$ is weakly $n-$nil clean if and only if $d = p^t(p-1)$ or $d = \frac{p^t(p-1)}{2}$ for some $0 \leq t \leq k-1$.

(2) $\mathbb{Z}_{p^k}$ is weakly $(x^n+x)-$nil clean if and only if $d=\frac{p^t(p-1)}{2}$ for some $0\leq t\leq k-1$.

Proof. (1) $\Leftarrow)$: Let $0\leq t\leq k-1$. If $d=p^t(p-1)$, then $\mathbb{Z}_{p^k}$ is (weakly) $n-$nil clean by Theorem 1. Suppose $d=\frac{p^t(p-1)}{2}$, then for any solution a of $x^{n-1}\equiv 1(\operatorname{mod} p^k)$, we have $a^{\frac{p-1}{2}}\equiv(a^{\frac{(p-1)}{2}})^{p^t}=a^{\frac{p^t(p-1)}{2}}\equiv 1(\operatorname{mod} p)$. Clearly, the congruence $x^{\frac{p-1}{2}}\equiv 1(\operatorname{mod} p)$ has $\frac{p-1}{2}$ solutions. By Lemma 2, those $\frac{p-1}{2}$ solutions correspond uniquely to $\frac{p-1}{2}$ solutions of $x^{\frac{p^t(p-1)}{2}}\equiv 1(\operatorname{mod} p^k)$. Let T_1 (respectivelyT_2) be the set of all sums (respectively, differences) of each of the $\frac{p-1}{2}$ solutions of $x^{\frac{p^t(p-1)}{2}}\equiv 1(\operatorname{mod} p^k)$ and each nilpotent in $\mathbb{Z}_{p^k}$. By imitating the proof of Theorem 1, we can see that T_1 (respectivelyT_2) has $(\frac{p-1}{2})p^{k-1}$ distinct elements. Moreover, if $a^{\frac{p^t(p-1)}{2}}\equiv 1(\operatorname{mod} p^k)$ and $b\in N(\mathbb{Z}_{p^k})$ such that $b+a=b-a$, then $2a=0$ which is a contradiction. Thus, $N(\mathbb{Z}_{p^k})\cup T_1\cup T_2$ contains exactly $(2(\frac{p-1}{2})+1)p^{k-1}=p^k$ distinct elements and so $\mathbb{Z}_{p^k}$ is weakly $n-$nil clean.

$\Rightarrow)$: Suppose $d\neq p^t(p-1)$ and $d\neq\frac{p^t(p-1)}{2}$ for all $0\leq t\leq k-1$. Then $d=mp^t$ for some $m\mid p-1$ where $m\neq p-1$. Hence, either $m=\frac{p-1}{2}$ or $m\lneqq\frac{p-1}{2}$. If $m=\frac{p-1}{2}$, then we get a contradiction. Suppose $m\lneqq\frac{p-1}{2}$ and consider $x^{mp^t}\equiv 1(\operatorname{mod} p^k)$. Then $x^m\equiv(x^m)^{p^t}\equiv 1(\operatorname{mod} p)$ has at most m solutions. Since $m\lneqq\frac{p-1}{2}$, then similar to the above argument, the set of all sums or difference of each nilpotent and each solution of x^n-x will not cover $\mathbb{Z}_{p^k}$. Thus, $\mathbb{Z}_{p^k}$ is not weakly $n-$nil clean.

(2) $\Rightarrow)$: If $d=p^t(p-1)$ for some $0\leq t\leq k-1$, then $x^{n-1}\equiv -1(\operatorname{mod} p^k)$ has no solution and so $\mathbb{Z}_{p^k}$ is not weakly (x^n+x)-nil clean. Suppose $d=mp^t$ where $0\leq t\leq k-1$, $m\neq p-1$ and $m\mid p-1$. If $m\lneqq\frac{p-1}{2}$, then similar to the proof of (1), $\mathbb{Z}_{p^k}$ is also not weakly (x^n+x)-nil clean. Hence, we must have $m=\frac{p-1}{2}$ and $d=\frac{p^t(p-1)}{2}$ for some $0\leq t\leq k-1$.

$\Leftarrow)$: Suppose $d=\frac{p^t(p-1)}{2}$then clearly $x^{\frac{p-1}{2}}\equiv(x^{\frac{p-1}{2}})^{p^t}\equiv -1(\operatorname{mod} p)$ has $\frac{p-1}{2}$ solutions each of which corresponds uniquely to a solution of $x^{\frac{p^t(p-1)}{2}}\equiv -1(\operatorname{mod} p^k)$. Define T_1 and T_2 as in (1) for the congruence $x^{\frac{p^t(p-1)}{2}}\equiv -1(\operatorname{mod} p^k)$, we can similarly see that $N(\mathbb{Z}_{p^k})\cup T_1\cup T_2$ contains exactly p^k distinct elements and so $\mathbb{Z}_{p^k}$ is weakly (x^n+x)-nil clean. □

Corollary 5. *Let $n,k\in\mathbb{N}$, p be any odd prime integer and $d=\gcd(n-1,p^{k-1}(p-1))$. Then $\mathbb{Z}_{p^k}$ is weakly $n-$nil clean that is not $n-$nil clean if and only if $d=\frac{p^t(p-1)}{2}$ for some $0\leq t\leq k-1$.*

For example $\mathbb{Z}_{5^k}$ is a weakly $3-$nil clean that is not $3-$nil clean for any $k\in\mathbb{N}$.

Now, we can use Theorem 3 and Proposition 2 to prove the following corollary.

Corollary 6. *Let $m,n,k\in\mathbb{N}$ and write $m=p_1^{r_1}p_2^{r_2}...p_t^{t}$ where $p_1,p_2,...,p_t$ are distinct prime integers. Then the ring $\mathbb{Z}_m$ is weakly $n-$nil clean if and only if there* *is at most* $1 \leq j \leq t$ *such that for some* $1 \leq l_j \leq r_j - 1$ $\gcd(n-1, p_j^{r_j-1}(p_j - 1)) = p_j^{l_j}(p_j - 1)$ *or* $\frac{p_j^{l_j}(p_j-1)}{2}$ *and* $\gcd(n-1, p_i^{r_i-1}(p_i - 1)) = p_i^{l_i}(p_i - 1)$ *for some* $1 \leq l_i \leq r_i - 1$ *for all* $i \neq j$*.*

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