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SOME NEW SIMPSON TYPE INEQUALITIES FOR THE p-CONVEX AND p-CONCAVE FUNCTIONS

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Abstract. In this paper, we establish some new Simpson type inequalities for the class of functions whose derivatives in absolute values at certain powers are p-convex and p-concave.

1. INTRODUCTION

A function $f : I \subseteq \mathbb{R} \to \mathbb{R}$ is said to be convex if the inequality

$$f\left(tx + (1-t)y\right) \leq tf(x) + (1-t)f(y)$$

is valid for all $x, y \in I$ and $t \in [0,1]$. If this inequality reverses, then f is said to be concave on interval $I \neq \emptyset$. This definition is well known in the literature.

It is well known that theory of convex sets and convex functions play an important role in mathematics and the other pure and applied sciences. In recent years, the concept of convexity has been extended and generalized in various directions using novel and innovative techniques. For some inequalities, generalizations and applications concerning convexity see [1, 2, 4, 5, 6, 16, 20].

In [9], the author gave definition harmonically convex and concave functions as follow.

Definition 1. *Let $I \subset \mathbb{R}\backslash\{0\}$ be a real interval. A function $f : I \to \mathbb{R}$ is said to be harmonically convex, if*

$$f\left(\frac{xy}{tx + (1-t)y}\right) \leq tf(y) + (1-t)f(x)$$

for all $x, y \in I$ and $t \in [0,1]$. If this inequality is reversed, then f is said to be harmonically concave.

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Definition 2. *Let $I \subset (0,\infty)$ be a real interval and $p \in \mathbb{R}\backslash\{0\}$. A function $f : I \to \mathbb{R}$ is said to be a p-convex function, if*

$$f\left(\left[tx^p + (1-t)y^p\right]^{1/p}\right) \leq tf(x) + (1-t)f(y)$$

for all $x, y \in I$ and $t \in [0,1]$. If this inequality is reversed, then f is said to be p-concave.

According to Definition 2, It can be easily seen that for $p = 1$ and $p = -1$, p-convexity reduces to ordinary convexity and harmonically convexity of functions defined on $I \subset (0,\infty)$, respectively.

Hermite-Hadamard inequality for the p-convex function is following:

Theorem 1. *Let $f : I \subset (0,\infty) \to \mathbb{R}$ be a p-convex function, $p \in \mathbb{R}\backslash\{0\}$, and $a, b \in I$ with $a < b$. If $f \in L[a,b]$ then we have*

$$f\left(\left[\frac{a^p + b^p}{2}\right]^{\frac{1}{p}}\right) \leq \frac{p}{b^p - a^p}\int_a^b \frac{f(x)}{x^{1-p}}dx \leq \frac{f(a)+f(b)}{2}.$$

These inequalities are sharp [5, 8]. If these inequalities are reversed, then f is said to be p-concave.

Many papers have been written by a number of mathematicians concerning inequalities for different classes of harmonically convex and p-convex functions see for instance the recent papers [3, 7, 8, 9, 10, 11, 12, 17, 18, 19, 21, 22, 24] and the references within these papers.

The following integral inequality, named Simpson's integral inequality, is one of the best known results in the literature.

Theorem 2. *(Simpson's Integral Inequality). Let $f : I = [a,b] \subset \mathbb{R} \to \mathbb{R}$ be a four time continuously differentiable on I°, where I° is the interior of I and $\left\|f^{(4)}\right\|_\infty < \infty$. Then*

$$\left|\frac{1}{3}\left[\frac{f(a)+f(b)}{2} + 2f\left(\frac{a+b}{2}\right)\right] - \frac{1}{b-a}\int_a^b f(x)dx\right| \leq \frac{1}{2880}\left\|f^{(4)}\right\|_\infty (b-a)^4.$$

There are substantial literature on Simpson type integral inequalities. Here we mention the result of [13, 14, 15] and the corresponding references cited therein.

Throughout this paper we will use the following notations. Let $0<a<b$ and $p\in\mathbb{R}\backslash\{0\}$.

$$\begin{aligned}
A_p &= A_p(a,b)=\frac{a^p+b^p}{2},\ A_1=A=A(a,b)=\frac{a+b}{2}\\
M_p &= M_p(a,b)=A_p^{\frac{1}{p}}=\left[\frac{a^p+b^p}{2}\right]^{\frac{1}{p}},\ H=H(a,b)=\frac{2ab}{a+b}\\
I_t(x,A_p;u,v) &= \frac{\left|t-\frac{1}{3}\right|^u}{\left[(1-t)x^p+tA_p\right]^{v-\frac{v}{p}}},\\
J_t(x,A_p;u,v) &= \frac{\left|t-\frac{1}{3}\right|^u(1-t)}{\left[(1-t)x^p+tA_p\right]^{v-\frac{v}{p}}},\\
K_t(x,A_p;u,v) &= \frac{\left|t-\frac{1}{3}\right|^u t}{\left[(1-t)x^p+tA_p\right]^{v-\frac{v}{p}}}.
\end{aligned}$$

where $t\in[0,1]$ and $u,v\geq 0$.

2. MAIN RESULTS

In this section, we will use the following Lemma for we obtain the main results:

Lemma 1. *Let $f:I\subset(0,\infty)\to\mathbb{R}$ be a differentiable mapping on I° (interior of I) and $a,b\in I^\circ$ with $a<b$ and $p\in\mathbb{R}\backslash\{0\}$. If $f'\in L[a,b]$, then the following equality holds:*

$$\begin{aligned}
&\frac{1}{6}\left[f(a)+4f(M_p)+f(b)\right]-\frac{p}{b^p-a^p}\int_a^b\frac{f(x)}{x^{1-p}}dx\\
=\ &\frac{b^p-a^p}{4p}\left[\int_0^1\frac{t-\frac{1}{3}}{\left[(1-t)a^p+tA_p\right]^{1-\frac{1}{p}}}f'\left(\left[(1-t)a^p+tA_p\right]^{\frac{1}{p}}\right)dt\right.\\
&\left.+\int_0^1\frac{t-\frac{2}{3}}{\left[(1-t)A_p+tb^p\right]^{1-\frac{1}{p}}}f'\left(\left[(1-t)A_p+tb^p\right]^{\frac{1}{p}}\right)dt\right].
\end{aligned}$$

Proof. Firstly, let's calculate the following integral:

$$\begin{aligned}
&\frac{b^p-a^p}{4p}\left[\int_0^1\frac{t-\frac{1}{3}}{\left[(1-t)a^p+tA_p\right]^{1-\frac{1}{p}}}f'\left(\left[(1-t)a^p+tA_p\right]^{\frac{1}{p}}\right)dt\right.\\
&\left.+\int_0^1\frac{t-\frac{2}{3}}{\left[(1-t)A_p+tb^p\right]^{1-\frac{1}{p}}}f'\left(\left[(1-t)A_p+tb^p\right]^{\frac{1}{p}}\right)dt\right].
\end{aligned}$$

For shortness, we will use the notations

$$\begin{aligned} I_1 &= \int_0^1 \left(t-\frac{1}{3}\right) df\left(\left[(1-t)\,a^p+tA_p\right]^{\frac{1}{p}}\right), \\ I_2 &= \int_0^1 \left(t-\frac{2}{3}\right) df\left(\left[(1-t)\,A_p+tb^p\right]^{\frac{1}{p}}\right). \end{aligned}$$

Using the partial integration method and the method of changing variables respectively for the integrals I_1 and I_2 as following, we get

$$\begin{aligned} I_1 &= \int_0^1 \left(t-\frac{1}{3}\right) df\left(\left[(1-t)\,a^p+tA_p\right]^{\frac{1}{p}}\right) \\ &= \left(t-\frac{1}{3}\right) f\left(\left[(1-t)\,a^p+tA_p\right]^{\frac{1}{p}}\right)\Big|_0^1 - \int_0^1 f\left(\left[(1-t)\,a^p+tA_p\right]^{\frac{1}{p}}\right) dt \\ &= \frac{2}{3}f\left(M_p\right)+\frac{1}{3}f\left(a\right)-\frac{2p}{b^p-a^p}\int_a^{A_p}\frac{f\left(x\right)}{x^{1-p}}dx \qquad (2.1) \end{aligned}$$

$$\begin{aligned} I_2 &= \int_0^1 \left(t-\frac{2}{3}\right) df\left(\left[(1-t)\,A_p+tb^p\right]^{\frac{1}{p}}\right) \\ &= \left(t-\frac{2}{3}\right) f\left(\left[(1-t)\,A_p+tb^p\right]^{\frac{1}{p}}\right)\Big|_0^1 - \int_0^1 f\left(\left[(1-t)\,A_p+tb^p\right]^{\frac{1}{p}}\right) dt \\ &= \frac{1}{3}f\left(b\right)+\frac{2}{3}f\left(M_p\right)-\frac{2p}{b^p-a^p}\int_{A_p}^{b}\frac{f\left(x\right)}{x^{1-p}}dx. \qquad (2.2) \end{aligned}$$

Summing up side by side (2.1) and (2.2), we have

$$\begin{aligned} I_1+I_2 &= \frac{1}{3}\left[f(a)+f(b)\right]+\frac{4}{3}f\left(M_p\right)-\frac{2p}{b^p-a^p}\int_a^b\frac{f\left(x\right)}{x^{1-p}}dx, \\ \frac{I_1+I_2}{2} &= \frac{1}{6}\left[f\left(a\right)+4f\left(M_p\right)+f\left(b\right)\right]-\frac{p}{b^p-a^p}\int_a^b\frac{f\left(x\right)}{x^{1-p}}dx. \end{aligned}$$

□

Theorem 3. *Let $f : I \subset (0,\infty) \to \mathbb{R}$ be a differentiable mapping on I° (the interior of I) and $a, b \in I^\circ$ with $a < b$ and $p \in \mathbb{R}\setminus\{0\}$. If $f' \in L[a,b]$ and $|f'|^q$ is p-convex on I for $q \geq 1$, then the following inequality holds:*

$$\begin{aligned} &\left|\frac{1}{6}\left[f\left(a\right)+4f\left(M_p\right)+f\left(b\right)\right]-\frac{p}{b^p-a^p}\int_a^b\frac{f\left(x\right)}{x^{1-p}}dx\right| \\ \leq\ & \frac{b^p-a^p}{4p}\left[C_p\left(a,b\right)\right]^{1-\frac{1}{q}}\left[\left|f'\left(a\right)\right|^q D_p\left(a,b\right)+\left|f'\left(M_p\right)\right|^q E_p\left(a,b\right)\right]^{\frac{1}{q}} \\ &+\frac{b^p-a^p}{4p}\left[F_p\left(a,b\right)\right]^{1-\frac{1}{q}}\left[\left|f'\left(M_p\right)\right|^q G_p\left(a,b\right)+\left|f'\left(b\right)\right|^q H_p\left(a,b\right)\right]^{\frac{1}{q}} \end{aligned}$$

where

$$
\begin{aligned}
C_p(a,b) &= \int_0^1 I_t(a,A_p;1,1)\,dt, \quad D_p(a,b)=\int_0^1 J_t(a,A_p;1,1)\,dt, \\
E_p(a,b) &= \int_0^1 K_t(a,A_p;1,1)\,dt, \quad F_p(a,b)=\int_0^1 I_{1-t}(b,A_p;1,1)\,dt, \\
G_p(a,b) &= \int_0^1 K_{1-t}(b,A_p;1,1)\,dt, \quad H_p(a,b)=\int_0^1 J_{1-t}(b,A_p;1,1)\,dt.
\end{aligned}
$$

Proof. Using Lemma 1 and the power mean inequality, we have

$$
\begin{aligned}
&\left|\frac{1}{6}\left[f(a)+4f(M_p)+f(b)\right]-\frac{p}{b^p-a^p}\int_a^b \frac{f(x)}{x^{1-p}}dx\right| \\
\leq\; & \frac{b^p-a^p}{4p}\left[\int_0^1 \frac{\left|t-\frac{1}{3}\right|}{\left[(1-t)a^p+tA_p\right]^{1-\frac{1}{p}}}\left|f'\left(\left[(1-t)a^p+tA_p\right]^{\frac{1}{p}}\right)\right|dt\right] \\
&+\frac{b^p-a^p}{4p}\left[\int_0^1 \frac{\left|t-\frac{2}{3}\right|}{\left[(1-t)A_p+tb^p\right]^{1-\frac{1}{p}}}\left|f'\left(\left[(1-t)A_p+tb^p\right]^{\frac{1}{p}}\right)\right|dt\right] \\
\leq\; & \frac{b^p-a^p}{4p}\left(\int_0^1 \frac{\left|t-\frac{1}{3}\right|}{\left[(1-t)a^p+tA_p\right]^{1-\frac{1}{p}}}dt\right)^{1-\frac{1}{q}} \\
&\times\left(\int_0^1 \frac{\left|t-\frac{1}{3}\right|}{\left[(1-t)a^p+tA_p\right]^{1-\frac{1}{p}}}\left|f'\left(\left[(1-t)a^p+tA_p\right]^{\frac{1}{p}}\right)\right|^q\right)^{\frac{1}{q}}dt \\
&+\frac{b^p-a^p}{4p}\left(\int_0^1 \frac{\left|t-\frac{2}{3}\right|}{\left[(1-t)A_p+tb^p\right]^{1-\frac{1}{p}}}dt\right)^{1-\frac{1}{q}} \\
&\times\left(\int_0^1 \frac{\left|t-\frac{2}{3}\right|}{\left[(1-t)A_p+tb^p\right]^{1-\frac{1}{p}}}\left|f'\left(\left[(1-t)A_p+tb^p\right]^{\frac{1}{p}}\right)\right|^q\right)^{\frac{1}{q}}dt \\
\leq\; & \frac{b^p-a^p}{4p}\left(\int_0^1 I_t(a,A_p;1,1)\,dt\right)^{1-\frac{1}{q}} \\
&\times\left(\int_0^1 \frac{\left|t-\frac{1}{3}\right|\left[(1-t)\left|f'(a)\right|^q+t\left|f'(M_p)\right|^q\right]}{\left[(1-t)a^p+tA_p\right]^{1-\frac{1}{p}}}\right)^{\frac{1}{q}}dt \\
&+\frac{b^p-a^p}{4p}\left(\int_0^1 I_{1-t}(b,A_p;1,1)\,dt\right)^{1-\frac{1}{q}} \\
&\times\left(\int_0^1 \frac{\left|t-\frac{2}{3}\right|\left[(1-t)\left|f'(M_p)\right|^q+t\left|f'(b)\right|^q\right]}{\left[(1-t)A_p+tb^p\right]^{1-\frac{1}{p}}}\right)^{\frac{1}{q}}dt
\end{aligned}
$$

$$
\begin{aligned}
= \quad & \frac{b^p-a^p}{4p}\left[\int_0^1 I_t\left(a,A_p;1,1\right)dt\right]^{1-\frac{1}{q}} \\
& \times\left[\left|f'\left(a\right)\right|^q\int_0^1 J_t\left(a,A_p;1,1\right)dt+\left|f'\left(M_p\right)\right|^q\int_0^1 K_t\left(a,A_p;1,1\right)dt\right]^{\frac{1}{q}} \\
& +\frac{b^p-a^p}{4p}\left[\int_0^1 I_{1-t}\left(b,A_p;1,1\right)dt\right]^{1-\frac{1}{q}} \\
& \times\left[\left|f'\left(M_p\right)\right|^q\int_0^1 K_{1-t}\left(b,A_p;1,1\right)dt+\left|f'\left(b\right)\right|^q\int_0^1 J_{1-t}\left(b,A_p;1,1\right)dt\right]^{\frac{1}{q}} \\
\leq \quad & \frac{b^p-a^p}{4p}\left[C_p\left(a,b\right)\right]^{1-\frac{1}{q}}\left[\left|f'\left(a\right)\right|^q D_p\left(a,b\right)+\left|f'\left(M_p\right)\right|^q E_p\left(a,b\right)\right]^{\frac{1}{q}} \\
& +\frac{b^p-a^p}{4p}\left[F_p\left(a,b\right)\right]^{1-\frac{1}{q}}\left[\left|f'\left(M_p\right)\right|^q G_p\left(a,b\right)+\left|f'\left(b\right)\right|^q H_p\left(a,b\right)\right]^{\frac{1}{q}}.
\end{aligned}
$$

This completes the proof of theorem. □

Corollary 1. *Under conditions of Theorem 3*

i. If we take $p=1$, then we obtain the following inequality for convex function:

$$
\begin{aligned}
& \left|\frac{1}{6}\left[f\left(a\right)+4f\left(\frac{a+b}{2}\right)+f\left(b\right)\right]-\frac{1}{b-a}\int_a^b f\left(x\right)dx\right| \\
\leq \quad & \frac{b-a}{4}\left(\frac{5}{18}\right)^{1-\frac{1}{q}}\left[\frac{8}{81}\left|f'\left(a\right)\right|^q+\frac{29}{162}\left|f'\left(\frac{a+b}{2}\right)\right|^q\right]^{\frac{1}{q}} \\
& +\frac{b-a}{4}\left(\frac{5}{18}\right)^{1-\frac{1}{q}}\left[\frac{29}{162}\left|f'\left(\frac{a+b}{2}\right)\right|^q+\frac{8}{81}\left|f'\left(b\right)\right|^q\right]^{\frac{1}{q}}
\end{aligned}
$$

which is the same of the inequality [6, Corollary 10] *for $s=1$.*

ii. If we take $p=-1$, then we obtain the following inequality for harmonically convex function:

$$
\begin{aligned}
& \left|\frac{1}{6}\left[f\left(a\right)+4f\left(\frac{2ab}{a+b}\right)+f\left(b\right)\right]-\frac{ab}{b-a}\int_a^b\frac{f\left(x\right)}{x^2}dx\right| \\
\leq \quad & \frac{b-a}{4ab}\left[C_{-1}\left(a,b\right)\right]^{1-\frac{1}{q}}\left[\left|f'\left(a\right)\right|^q D_{-1}\left(a,b\right)+\left|f'\left(\frac{2ab}{a+b}\right)\right|^q E_{-1}\left(a,b\right)\right]^{\frac{1}{q}} \\
& +\frac{b-a}{4ab}\left[F_{-1}\left(a,b\right)\right]^{1-\frac{1}{q}}\left[\left|f'\left(\frac{2ab}{a+b}\right)\right|^q G_{-1}\left(a,b\right)+\left|f'\left(b\right)\right|^q H_{-1}\left(a,b\right)\right]^{\frac{1}{q}}.
\end{aligned}
$$

Theorem 4. *Let $f:I\subset(0,\infty)\rightarrow\mathbb{R}$ be a differentiable mapping on I° (the interior of I) and $a,b\in I^\circ$ with $a<b$ and $p\in\mathbb{R}\setminus\{0\}$. If $f'\in L[a,b]$ and $\left|f'\right|^q$ is p-convex*

on I for $q>1$, $\frac{1}{r}+\frac{1}{q}=1$, then

$$\begin{aligned}&\left|\frac{1}{6}\left[f(a)+4f(M_p)+f(b)\right]-\frac{p}{b^p-a^p}\int_a^b\frac{f(x)}{x^{1-p}}dx\right|\\ \leq\ &\frac{b^p-a^p}{4p}\left[N_{p,r}^{\frac{1}{r}}(a,b)\,A^{\frac{1}{q}}\left(\left|f'(a)\right|^q,\left|f'(M_p)\right|^q\right)+O_{p,r}^{\frac{1}{r}}(a,b)\,A^{\frac{1}{q}}\left(\left|f'(M_p)\right|^q,\left|f'(b)\right|^q\right)\right],\end{aligned}$$

where

$$\begin{aligned}N_{p,r}(a,b)&=\int_0^1 I_t(a,A_p;r,r)\,dt,\\ O_{p,r}(a,b)&=\int_0^1 I_{1-t}(b,A_p;r,r)\,dt.\end{aligned}$$

Proof. From Lemma 1, Hölder's integral inequality and the p-convexity of $|f'|^q$ on $[a,b]$, we have,

$$\begin{aligned}&\left|\frac{1}{6}\left[f(a)+4f(M_p)+f(b)\right]-\frac{p}{b^p-a^p}\int_a^b\frac{f(x)}{x^{1-p}}dx\right|\\ \leq\ &\frac{b^p-a^p}{4p}N_{p,r}^{\frac{1}{r}}(a,b)\left(\int_0^1\left|f'\left(\left((1-t)a^p+tA_p\right)^{\frac{1}{p}}\right)\right|^q dt\right)^{\frac{1}{q}}\\ &+\frac{b^p-a^p}{4p}O_{p,r}^{\frac{1}{r}}(a,b)\left(\int_0^1\left|f'\left(\left((1-t)A_p+tb^p\right)^{\frac{1}{p}}\right)\right|^q dt\right)^{\frac{1}{q}}\\ \leq\ &\frac{b^p-a^p}{4p}\left[N_{p,r}^{\frac{1}{r}}(a,b)\left(\int_0^1\left((1-t)\left|f'(a)\right|^q+t\left|f'(M_p)\right|^q\right)dt\right)^{\frac{1}{q}}\right.\\ &\left.+\,O_{p,r}^{\frac{1}{r}}(a,b)\left(\int_0^1\left((1-t)\left|f'(M_p)\right|^q+t\left|f'(b)\right|^q\right)dt\right)^{\frac{1}{q}}\right]\\ =\ &\frac{b^p-a^p}{4p}\left[N_{p,r}^{\frac{1}{r}}(a,b)\left(\frac{\left|f'(a)\right|^q+\left|f'(M_p)\right|^q}{2}\right)+O_{p,r}^{\frac{1}{r}}(a,b)\left(\frac{\left|f'(M_p)\right|^q+\left|f'(b)\right|^q}{2}\right)\right]\\ =\ &\frac{b^p-a^p}{4p}\left[N_{p,r}^{\frac{1}{r}}(a,b)\,M_q\left(\left|f'(a)\right|,\left|f'(M_p)\right|\right)+O_{p,r}^{\frac{1}{r}}(a,b)\,M_q\left(\left|f'(M_p)\right|,\left|f'(b)\right|\right)\right].\end{aligned}$$

This completes the proof of theorem. □

Corollary 2. *Under conditions of Theorem 4,*

i. If we take $p=1$, then we obtain the following inequality for convex function:

$$\begin{aligned}&\left|\frac{1}{6}\left[f(a)+4f\left(\frac{a+b}{2}\right)+f(b)\right]-\frac{1}{b-a}\int_a^b f(x)\,dx\right|\\ \leq\ &\frac{b-a}{12}\left[\frac{1+2^{r+1}}{3(r+1)}\right]^{\frac{1}{r}}\left[M_q\left(\left|f'(a)\right|,\left|f'\left(\frac{a+b}{2}\right)\right|\right)+M_q\left(\left|f'\left(\frac{a+b}{2}\right)\right|,\left|f'(b)\right|\right)\right]\end{aligned}$$

which is the same as the inequality in [22, Theorem 8] *for* $s = 1$.

ii. If we take $p = -1$*, then we obtain the following inequality for harmonically convex function:*

$$\begin{aligned} & \left| \frac{1}{6}\left[f(a) + 4f\left(\frac{2ab}{a+b}\right) + f(b)\right] - \frac{ab}{b-a}\int_a^b \frac{f(x)}{x^2}dx \right| \\ \leq \quad & \frac{b-a}{4ab}\left[N_{-1,r}^{\frac{1}{r}}(a,b)\, M_q\left(\left|f'(a)\right|, \left|f'(H)\right|\right) + O_{-1,r}^{\frac{1}{r}}(a,b)\, M_q\left(\left|f'(H)\right|, \left|f'(b)\right|\right)\right]. \end{aligned}$$

Theorem 5. *Let* $f : I \subset (0,\infty) \to \mathbb{R}$ *be a differentiable mapping on* I° *(the interior of* I*) and* $a, b \in I^{\circ}$ *with* $a < b$ *and* $p \in \mathbb{R}\backslash\{0\}$*. If* $f' \in L[a,b]$ *and* $|f'|^q$ *is* p*-convex on* I *for* $q > 1$, $\frac{1}{r} + \frac{1}{q} = 1$*, then*

$$\begin{aligned} & \left| \frac{1}{6}\left[f(a) + 4f(M_p) + f(b)\right] - \frac{p}{b^p - a^p}\int_a^b \frac{f(x)}{x^{1-p}}dx \right| \\ \leq \quad & \frac{b^p - a^p}{12p}\left[\frac{1+2^{r+1}}{3(r+1)}\right]^{\frac{1}{r}} \left\{ \left[Q_{p,q}(a,b)\left|f'(a)\right|^q + R_{p,q}(a,b)\left|f'(M_p)\right|^q\right]^{\frac{1}{q}} \right. \\ & \left. + \left[S_{p,q}(a,b)\left|f'(b)\right|^q + T_{p,q}(a,b)\left|f'(M_p)\right|^q\right]^{\frac{1}{q}} \right\}, \end{aligned}$$

where

$$\begin{aligned} Q_{p,q}(a,b) &= \int_0^1 J_t(a, A_p; 0, q)\, dt, \qquad S_{p,q}(a,b) = \int_0^1 K_{1-t}(b, A_p; 0, q)\, dt, \\ R_{p,q}(a,b) &= \int_0^1 K_t(a, A_p; 0, q)\, dt, \qquad T_{p,q}(a,b) = \int_0^1 J_{1-t}(b, A_p; 0, q)\, dt. \end{aligned}$$

Proof. From Lemma 1, Hölder's integral inequality and the p-convexity of $|f'|^q$ on $[a,b]$, we obtain,

$$\begin{aligned}
&\left|\frac{1}{6}\left[f(a)+4f(M_p)+f(b)\right]-\frac{p}{b^p-a^p}\int_a^b \frac{f(x)}{x^{1-p}}dx\right| \\
\leq\ & \frac{b^p-a^p}{4p}\int_0^1\left|t-\frac{1}{3}\right|\left|\frac{1}{\left[(1-t)a^p+tA_p\right]^{1-\frac{1}{p}}}f'\left(\left[(1-t)a^p+tA_p\right]^{\frac{1}{p}}\right)\right|dt \\
&+\frac{b^p-a^p}{4p}\int_0^1\left|t-\frac{2}{3}\right|\left|\frac{1}{\left[(1-t)A_p+tb^p\right]^{1-\frac{1}{p}}}f'\left(\left[(1-t)A_p+tb^p\right]^{\frac{1}{p}}\right)\right|dt \\
\leq\ & \frac{b^p-a^p}{12p}\left[\frac{1+2^{r+1}}{3(r+1)}\right]^{\frac{1}{r}}\left(\int_0^1\frac{1}{\left[(1-t)a^p+tA_p\right]^{q-\frac{q}{p}}}\left|f'\left(\left[(1-t)a^p+tM_p^p\right]^{\frac{1}{p}}\right)\right|^q dt\right)^{\frac{1}{q}} \\
&+\frac{b^p-a^p}{12p}\left[\frac{1+2^{r+1}}{3(r+1)}\right]^{\frac{1}{r}}\left(\int_0^1\frac{1}{\left[(1-t)A_p+tb^p\right]^{q-\frac{q}{p}}}\left|f'\left(\left[(1-t)M_p^p+tb^p\right]^{\frac{1}{p}}\right)\right|^q dt\right)^{\frac{1}{q}} \\
\leq\ & \frac{b^p-a^p}{12p}\left[\frac{1+2^{r+1}}{3(r+1)}\right]^{\frac{1}{r}}\left\{\left[Q_{p,q}(a,b)\left|f'(a)\right|^q+R_{p,q}(a,b)\left|f'(M_p)\right|^q\right]^{\frac{1}{q}}\right. \\
&\left.+\left[S_{p,q}(a,b)\left|f'(b)\right|^q+T_{p,q}(a,b)\left|f'(M_p)\right|^q\right]^{\frac{1}{q}}\right\}.
\end{aligned}$$

This completes the proof of theorem. □

Corollary 3. *Under conditions of Theorem 5,*

i. If we take $p=1$, then we obtain the following inequality for convex function:

$$\begin{aligned}
&\left|\frac{1}{6}\left[f(a)+4f\left(\frac{a+b}{2}\right)+f(b)\right]-\frac{1}{b-a}\int_a^b f(x)\,dx\right| \\
\leq\ & \frac{b-a}{12}\left[\frac{1+2^{r+1}}{3(r+1)}\right]^{\frac{1}{r}}\left[M_q\left(\left|f'(a)\right|,\left|f'\left(\frac{a+b}{2}\right)\right|\right)+M_q\left(\left|f'\left(\frac{a+b}{2}\right)\right|,\left|f'(b)\right|\right)\right]
\end{aligned}$$

which reduce the inequality in Corollary 2 (i).

ii. If we take $p=-1$, then we obtain the following inequality for harmonically convex function:

$$\begin{aligned}
&\left|\frac{1}{6}\left[f(a)+4f\left(\frac{2ab}{a+b}\right)+f(b)\right]-\frac{ab}{b-a}\int_a^b\frac{f(x)}{x^2}dx\right| \\
\leq\ & \frac{b-a}{12ab}\left[\frac{1+2^{r+1}}{3(r+1)}\right]^{\frac{1}{r}}\left\{\left[Q_{-1,q}(a,b)\left|f'(a)\right|^q+R_{-1,q}(a,b)\left|f'(H)\right|^q\right]^{\frac{1}{q}}\right. \\
&\left.+\left[S_{-1,q}(a,b)\left|f'(b)\right|^q+T_{-1,q}(a,b)\left|f'(H)\right|^q\right]^{\frac{1}{q}}\right\}.
\end{aligned}$$

Theorem 6. *Let $f : I \subset (0,\infty) \to \mathbb{R}$ be a differentiable mapping onI° (the interior of I) and $a, b \in I^{\circ}$ with $a < b$ and $p \in \mathbb{R} \setminus \{0\}$. If $f' \in L[a,b]$ and $|f'|^q$ is p-concave on I for $q > 1$, $\frac{1}{r} + \frac{1}{q} = 1$, then*

$$\begin{aligned} &\left| \frac{1}{6}\left[f(a) + 4f(M_p) + f(b) \right] - \frac{p}{b^p - a^p} \int_a^b \frac{f(x)}{x^{1-p}} dx \right| \\ \leq\ & \frac{b^p - a^p}{4p} \left[N_{p,r}^{\frac{1}{r}}(a,b) \left| f'\left(\left[\frac{3a^p + b^p}{4} \right]^{\frac{1}{p}} \right) \right| + O_{p,r}^{\frac{1}{r}}(a,b) \left| f'\left(\left[\frac{a^p + 3b^p}{4} \right]^{\frac{1}{p}} \right) \right| \right]. \end{aligned}$$

Proof. From Lemma 1, Hölder's integral inequality and the p-concavity of$|f'|^q$ on $[a,b]$, we have,

$$\begin{aligned} &\left| \frac{1}{6}\left[f(a) + 4f(M_p) + f(b) \right] - \frac{p}{b^p - a^p} \int_a^b \frac{f(x)}{x^{1-p}} dx \right| \\ \leq\ & \frac{b^p - a^p}{4p} N_{p,r}^{\frac{1}{r}}(a,b) \left(\int_0^1 \left| f'\left(\left((1-t)\, a^p + tA_p \right)^{\frac{1}{p}} \right) \right|^q dt \right)^{\frac{1}{q}} \\ & + \frac{b^p - a^p}{4p} O_{p,r}^{\frac{1}{r}}(a,b) \left(\int_0^1 \left| f'\left(\left((1-t)\, A_p + tb^p \right)^{\frac{1}{p}} \right) \right|^q dt \right)^{\frac{1}{q}} \\ \leq\ & \frac{b^p - a^p}{4p} \left[N_{p,r}^{\frac{1}{r}}(a,b) \left| f'\left(\left[\frac{3a^p + b^p}{4} \right]^{\frac{1}{p}} \right) \right| + O_{p,r}^{\frac{1}{r}}(a,b) \left| f'\left(\left[\frac{a^p + 3b^p}{4} \right]^{\frac{1}{p}} \right) \right| \right]. \end{aligned}$$

This completes the proof of theorem. □

Corollary 4. *Under conditions of Theorem 6,*

i. If we take $p = 1$, then we obtain the following inequality for concave function:

$$\begin{aligned} &\left| \frac{1}{6}\left[f(a) + 4f\left(\frac{a+b}{2} \right) + f(b) \right] - \frac{1}{b-a} \int_a^b f(x)\, dx \right| \\ \leq\ & \frac{b-a}{12} \left[\frac{1 + 2^{r+1}}{3(r+1)} \right]^{\frac{1}{r}} \left[\left| f'\left(\frac{3a+b}{4} \right) \right| + \left| f'\left(\frac{a+3b}{4} \right) \right| \right] \end{aligned}$$

This is the same of the inequality in [6, Corollary 28] *for $s = 1$.*

ii. If we take $p = -1$, then we obtain the following inequality for harmonically concave function:

$$\begin{aligned} &\left| \frac{1}{6}\left[f(a) + 4f\left(\frac{2ab}{a+b} \right) + f(b) \right] - \frac{ab}{b-a} \int_a^b \frac{f(x)}{x^2} dx \right| \\ \leq\ & \frac{b-a}{4ab} \left[N_{-1,r}^{\frac{1}{r}}(a,b) \left| f'\left(\frac{4ab}{a+3b} \right) \right| + O_{-1,r}^{\frac{1}{r}}(a,b) \left| f'\left(\frac{4ab}{3a+b} \right) \right| \right]. \end{aligned}$$

3. CONCLUSION

The paper deals with Simpson type inequalities for p-convex and p-concave functions. Firstly, we give a new identity for differentiable functions and get some new integral inequalities for the p-convex and p-concave functions by using this identity.

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