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PORTFOLIO OPTIMIZATION UNDER PARAMETER UNCERTAINTY USING THE RISK AVERSION FORMULA

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Abstract. The Markowitz portfolio optimization model has certain difficulties in practise since real data are rarely certain. The robust optimization is a recently developed method that is used to overcome the uncertainty situation. The technique has been recently suggested in the portfolio selection problems. In this study, two kinds of portfolio optimization problems are presented: (i) the risk aversion portfolio optimization problem based on the classical Markowitz framework, and (ii) the max-min counterpart problem based on the robust optimization framework. In the application, the two models are performed on a real-world data set obtained from BIST (Borsa Istanbul). Numerical results show that the objective function values of the classical solution and the robust solution are similar to each other. It can be said that the robust model, which works as well as the classical model in the uncertainty situations, can be used instead of the classical model and also that the optimal solution obtained in the uncertainty situation is robust to parameter perturbation.

1. INTRODUCTION

The main objective of the portfolio optimization problem is to choose the optimal portfolio with minimum variance from the set of all possible portfolios for any given level of expected return. Markowitz [22] formulated the first mathematical model for portfolio selection in the literature. After Markowitz, Sharpe [26] developed the Capital Asset Pricing Model (CAPM) and then Linter [19] and Mossin [24] used the Markowitz theory in their studies. In the literature, there are various other portfolio optimization methods developed in the context of the portfolio theory besides Markowitz, such as safety-first models, elliptical distributions, value at risk-based optimization, maximizing the performance measures EVA and RAROC and modelling the uncertainty of input parameters [10].

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The Markowitz mean-variance portfolio optimization is a well-known investment theory that is widely used in allocating the assets. Its biggest influence can be seen on the practice of portfolio management. The theory is focused on evaluating and managing the risks and returns of a portfolio of investments. This is highly advantageous as the resulting "optimized" portfolio will either have the same expected return with fewer risks than before or a higher expected return with the same level of risk. The Markowitz mean-variance optimization problem has several alternative formulations that are used in practical applications. One of these alternative formulations is using a risk aversion coefficient in the model, which is called the risk aversion formulation. This study handles the risk aversion formulation of the classical Markowitz model.

Although the Markowitz model is successful in the theory, there are various challenges of the model. The parameter uncertainty is an important issue in the optimization problems. In the Markowitz model, the uncertainty in the market parameters affects the optimal solution of the problem. Thus, the results cannot be reliable enough. There are numerous studies in the literature to overcome the difficulties of the Markowitz model: Chopra and Ziemba [8] studied the estimated parameters. Broadie and Chopra [6] used the estimation errors in their study. Chopra [7] and Frost and Savarino [12, 13] presented a method related to the portfolio weights. Chopra et al. [9] used the James-Stein estimator for the means, Klein and Bawa [16], Frost and Savarino [12], and Black and Litterman [5] used the Bayesian estimation of means and covariances [19].

An underlying assumption of Markowitz's model is that the precise estimates of μ_i and σ_{ij} have been obtained. Consequently, μ_i and σ_{ij} are treated as known constants; however, asset returns are variable. It is reasonable to conclude that a model which treats returns as known constants will produce a portfolio whose realized return is different from the optimal portfolio return given by the objective function value. In particular, when the realized asset returns are less than the estimates used to optimize the model, the realized portfolio return will be less than the optimal portfolio return given by the objective. Therefore, it is worthwhile exploring the alternative frameworks, such as the robust optimization, for application to the portfolio selection problem. Although the distributions of asset returns are uncertain, in the robust optimization framework, it may be asserted that $\mu$and σ, or both, belong to an uncertainty set, the bounds of which can be defined [15].

The aim of the method is to obtain a solution that is robust to the parameter uncertainty and estimation errors. In this framework, the ***robust counterpart*** of the original problem is handled. The robust problem is in fact the ***worst-case*** formulation of the original problem.

The first studies in the robust optimization framework are given in the studies of Ben-Tal and Nemirovski [2], [3], [4]. The first study handled the robust approach for linear programming. The other studies introduced the robust framework for convex programming. In these studies, it is assumed that the model parameters are unknown, but they are bounded and belong to the specific uncertainty sets defined by historical knowledge. The aim of the robust (worst-case) approach is to obtain the optimal solution of the model which is robust to the parameter uncertainty and the *worst-case* situation. The ***robust counterpart*** of the original problem is handled in the robust model.

Goldfarb and Iyengar handled various robust portfolio selection problems in their study, such as the robust mean-variance portfolio selection, the robust minimum variance problem, the robust maximum return problem, the robust maximum Sharpe Ratio problem, and the robust value at risk problem [14].

There are many latest references in the literature about the robust portfolio selection problem. Wang and Cheng [28] considered the robust portfolio selection problem which has a data uncertainty described by the (p, w)-norm in the objective function. Balbás A., Balbás B. and Balbás R. [1] handled portfolio selection problems under risk and ambiguity. Yu X. [29] developed a multi-period mean-variance model where the model parameters change according to a market with Markov random regime switching. Nalan G. and Canakoglu E. [25] considered a portfolio selection problem under temperature uncertainty. They introduced stochastic and robust portfolio optimization models using weather derivatives. Lotfi S., Salahi M. and Mehrdoust F. [20] used the robust optimization approach to address the ambiguity in the conditional value-at-risk minimization model.

The aim of this study is to introduce the risk aversion portfolio selection problem under the input parameter μ uncertainty. This problem is called the ($maxmin$) robust counterpart of the risk aversion problem. Moreover, it is aimed to obtain the optimal portfolio (the optimal solution of the robust problem) under this uncertainty and to compare the solution with the classical risk aversion solution.

In Inan [16] and Inan [17], the robust optimization approach is studied on the portfolio optimization problem. Numerical results showed that the classical optimal solution and the robust optimal solution gave similar values to the objective function. As a result, the optimal solution obtained in the uncertainty case is robust to the uncertainty case. The finding in the study is consistent with these studies.

The rest of this paper is organized as follows: In Section 2, the Markowitz portfolio optimization model and another alternative model, the risk aversion problem, is introduced. Section 3 presents the robust portfolio optimization method. The (max-min) robust counterpart of the problem is given. Finally, the max-min problem is converted into the classical maximum problem by the Lagrange method. In Section 4, a numerical example of the model with a real data set is handled. The data is taken from BIST (Borsa Istanbul). In Section 5, some conclusions in certain and uncertain situations are given.

2. MARKOWITZ MEAN-VARIANCE PORTFOLIO OPTIMIZATION PROBLEM

Harry Markowitz published his study and formed the basis for the ***mean-variance optimization*** "Portfolio Selection" in 1952. He suggested that investors should create the optimal portfolio based on the balance between the expected return and the risk. In the Markowitz portfolio model, the returns are defined as the mean vector, and the risk is defined as the variance of return. The model uses the optimization and probability methods together under uncertainty. The model comprises the return matrix, the mean vector and variance-covariance matrix components.

Suppose that an investor has a portfolio comprised of n risky assets, denoted as S_i The return of the security S_i is defined as R_i, and the weight of the i.security in the portfolio is defined as X_i.

The model can be created in two frameworks: (i) minimizing the risk of the portfolio for a certain level of expected return, (ii) maximizing the return of the portfolio for a certain level of risk.

The first model is given as,

$$\min X^t \Sigma X$$

$$\begin{aligned} \mu^t X &\geq \alpha \\ \sum_{i=1}^{n} X_i &= 1 \\ X_i &\geq 0, \quad i = 1, ..., n \end{aligned} \tag{2.1}$$

The second model is given as,

$$\max \mu^t X$$

$$\begin{aligned} X^t \Sigma X &\leq \beta \\ \sum_{i=1}^{n} X_i &= 1 \\ X_i &\geq 0, \quad i = 1, ..., n \end{aligned} \tag{2.2}$$

where α, β are constant, which are called the level degree. The descriptions of the model components are given as follows:

$(R_{k1}, ..., R_{kn})^t$ represents the n kinds of returns at time k $(k = 1, ..., m)$, where R_{ki} is the return of i.securities, $i = 1, ..., n$, $k = 1, ..., m$. The total data matrix is represented as,

$$\begin{bmatrix} R_{11} & ... & R_{1n} \\ \vdots & & \vdots \\ R_{m1} & ... & R_{mn} \end{bmatrix} \tag{2.3}$$

The return vector is denoted as $R = [R_1 ... R_n]^t$ in m period, it contains the expected value (mean) of each security. The expected vector of R is denoted as;

$$\mu = [\mu_1 .. \mu_n]^t$$

The input parameters μ and Σ are not certain. It is very difficult to estimate the correct values of these parameters. In the Markowitz model, the estimates of these parameters are used as follows:

$$\mu = [\mu_1 .. \mu_n]^t \tag{2.4}$$

$\mu = \left[\sum_{k=1}^{m} \frac{R_{k1}}{m} ... \sum_{k=1}^{m} \frac{R_{kn}}{m}\right]^t$ and the covariance matrix is given by,

$$\Sigma = \begin{bmatrix} \sigma_{11} & ... & \sigma_{1N} \\ \cdots & & \cdots \\ \sigma_{N1} & \cdots & \sigma_{NN} \end{bmatrix} \tag{2.5}$$

Here; $\sigma_{ij}$is the covariance between asset i and asset j.

The corresponding variance is given as,

$$\sigma_{ij}{}^2 = \sum_{k=1}^{m} \frac{(R_{ki} - \mu_i)\left(R_{kj} - \mu_j\right)}{m-1} \tag{2.6}$$

Thus, the random return vector R is represented by the (μ, Σ), [25].

There are two different definitions of R. One of them is the random vector. In finance applications, one should use the adjusted (from splits and dividends) stock prices to make calculations. However, it is difficult to obtain the adjusted stock prices from the splits and dividends, so only the closing prices are used in the study.

The alternative model that combines the risk and the return of the objective function can be created using the coefficient of risk aversion. The risk aversion formulation problem is defined as,

$$\max \left(\mu^{'} X - \lambda X^{'} \Sigma X\right)$$

$$X^{'} l = 1, \quad l = [1, 1, ..., 1] \tag{2.7}$$

where, λ is the risk aversion coefficient. When the investor is exposed to the uncertainty situation, the risk aversion coefficient can be used to reduce that uncertainty. If λ is large, the aversion to the risk is high. For example, the risk-averse investor might make an investment in treasury bonds that have low but guaranteed expected returns. Otherwise, if λ is small, the aversion to the risk is low. For example, the risk-loving investor might make an investment in stocks, the options of which have high expected returns but also high risks.

3. ROBUST PORTFOLIO OPTIMIZATION PROBLEM

In spite of the theoretical success of the mean-variance model, practitioners have shied away from this model. The solution of optimization problems is often very sensitive to perturbations in the parameters of the problem. Since the estimates of the market parameters are subject to statistical errors, they are very sensitive to the perturbations in the inputs. The results of the optimization problems may not be very reliable. There are a number of discussions on how to decrease or eliminate the possibility of using incorrect inputs for the optimization problem. Various aspects of this phenomenon have been extensively studied in the literature on portfolio selection.

Michaud [23] proposed to use the technique of resampling. In his study, he suggested resampling the input parameters from a confidence region and then averaging the cumulative portfolios that were obtained by each pair of sampling data. The main idea is that if resampling was performed enough times, the averaged optimal portfolio would be more stable and less sensitive to the perturbations in the inputs. But when the amount of assets becomes large, this method is not useful and efficient [21].

The robust optimization is the one of the aspects in the portfolio selection problems [14]. In the robust approach, the ***worst-case*** formulation of the original optimization problem, called the ***robust counterpart*** of the problem, is handled. The robust counterpart of the classical risk aversion model is used in this study.

In [11], the (*maxmin*) ***robust counterpart*** of the risk aversion model is given as

$$\max_{X} \min_{\mu \in U_\delta(\hat{\mu})} \left(\mu' X - \lambda X' \Sigma X \right)$$

$$X' l = 1, \quad l = [1, 1, ..., 1] \tag{3.1}$$

In the problem, it is assumed that the expected return vector $\mu$is unknown but belongs to the specific uncertainty set $U_\delta(\hat{\mu})$. Many special uncertainty sets are defined for the uncertain parameters in the literature. In this study, the uncertainty set for μ is taken as

$$U_\delta(\hat{\mu}) = \{\mu / (\mu - \hat{\mu})' (\Sigma_\mu)^{-1} (\mu - \hat{\mu}\ ) \leq \delta^2\} \tag{3.2}$$

where the parameters of the model are defined as follows:

$\hat{\mu}$:The estimated expected return vector

μ :The true expected return vector

$\Sigma_\mu = \frac{1}{T}\Sigma$, Estimation error covariance matrix

T : Return data observations for N assets.

δ : small number ($\delta > 0$)

The aim of the problem is to determine the weight vector X, which is robust to the uncertainty and the worst-case realization of the μ parameter.

For solving the robust (*maxmin*) problem easily, the problem is converted to the standard *maximum* optimization problem as follows:

Firstly, the uncertainty set is written as a constraint, then the problem can be written as

$$min_{\mu}\left(\mu^{'}X-\lambda X^{'}\Sigma X\right)$$

$$(\mu-\hat{\mu})^{'}(\Sigma_{\mu})^{-1}(\mu-\hat{\mu})\leq\delta^{2} \tag{3.3}$$

To solve this problem, the Lagrangian method can be used. The Lagrangian of the problem takes the form,

$$L\left(\mu,\gamma\right)=\mu^{'}X-\lambda X^{'}\Sigma X-\gamma\left(\delta^{2}-(\mu-\hat{\mu})^{'}(\Sigma_{\mu})^{-1}(\mu-\mu)\right) \tag{3.4}$$

The optimal values of μ and γ are obtained by the first order condition as

$$\mu^{*}=\hat{\mu}-\frac{1}{2\gamma}\Sigma_{\mu}X \tag{3.5}$$

$$\gamma^{*}=\frac{1}{2\delta}\sqrt{X'\Sigma_{\mu}X} \tag{3.6}$$

Finally, by substituting the expressions in the Lagrangian form, the robust problem is obtained as

$$\max\left(\mu^{'}X-\lambda X^{'}\Sigma X-\delta\sqrt{X'\Sigma_{\mu}X}\right)$$

$$X^{'}l=1. \tag{3.7}$$

4. APPLICATION

In this section, the robust portfolio selection approach, which was originally presented in the study by Fabozzi et al., is suggested. The data set is taken as the daily closing prices of nine securities that cycled in BIST 100 between 20.08.2013 and 20.08.2015. In the study, the daily stock price is chosen instead of the monthly stock price because the number of monthly stock price, which is 24 (for two years), may not be enough for the application.

The securities taken from the automotive sector belong to **Balat, Asuzu, Daos, Karsn, Tmsn, Froto, Toaso, Ttrak,** and **Otkar.** The returns of the securities were calculated according to the expression $ln\left(P_t/P_{t-1}\right)$ of the closing prices. Here;

P_t : Closing prices of t. day

The average vector μ, the variance covariance matrix Σ, and the estimation error covariance matrix Σ_M are calculated on the returns. Here;

$$\Sigma_M=\frac{1}{T}\Sigma$$

T :Return data observations for N assets

In this study, T is given as 502 days between the designated dates (20.08.2013–20.08.2015). The return vector μ, the Σ variance covariance matrix Σ and the estimation error covariance matrix Σ_μ are obtained as follows. Note that the values in the variance covariance matrix Σ_μ are multiplied by 1000.

$$\mu = \begin{bmatrix} 0.00000620 \\ -0.00157000 \\ 0.00088650 \\ 0.00018563 \\ 0.00063045 \\ 0.00069976 \\ 0.00043500 \\ 0.00091629 \\ 0.00054343 \end{bmatrix}$$

$$\Sigma = \begin{matrix} 0.000623 & 0.000047 & 0.000363 & 0.000214 & 0.000249 & 0.000291 & 0.000283 & 0.000238 & 0.000200 \\ 0.000047 & 0.000804 & 0.000076 & 0.000024 & 0.000119 & 0.000047 & 0.000102 & 0.000053 & 0.000039 \\ 0.000363 & 0.000076 & 0.000869 & 0.000306 & 0.000324 & 0.000338 & 0.000340 & 0.000396 & 0.000213 \\ 0.000214 & 0.000024 & 0.000306 & 0.000429 & 0.000146 & 0.000216 & 0.000226 & 0.000281 & 0.000137 \\ 0.000249 & 0.000119 & 0.000324 & 0.000146 & 0.000607 & 0.000254 & 0.000297 & 0.000188 & 0.000158 \\ 0.000291 & 0.000047 & 0.000338 & 0.000216 & 0.000254 & 0.000674 & 0.000337 & 0.000243 & 0.000213 \\ 0.000283 & 0.000102 & 0.000340 & 0.000226 & 0.000297 & 0.000337 & 0.000765 & 0.000239 & 0.000177 \\ 0.000238 & 0.000053 & 0.000396 & 0.000281 & 0.000188 & 0.000243 & 0.000239 & 0.000598 & 0.000149 \\ 0.000200 & 0.000039 & 0.000213 & 0.000137 & 0.000158 & 0.000213 & 0.000177 & 0.000149 & 0.000378 \end{matrix}$$

$$\Sigma_\mu = \begin{matrix} 0.001246 & 0.000093 & 0.000725 & 0.000429 & 0.000497 & 0.000582 & 0.000566 & 0.000476 & 0.000400 \\ 0.000093 & 0.001607 & 0.000152 & 0.000048 & 0.000239 & 0.000094 & 0.000204 & 0.000105 & 0.000077 \\ 0.000725 & 0.000152 & 0.001739 & 0.000611 & 0.000648 & 0.000676 & 0.000680 & 0.000791 & 0.000426 \\ 0.000429 & 0.000048 & 0.000611 & 0.000858 & 0.000293 & 0.000433 & 0.000451 & 0.000562 & 0.000275 \\ 0.000497 & 0.000239 & 0.000648 & 0.000293 & 0.001215 & 0.000507 & 0.000595 & 0.000376 & 0.000315 \\ 0.000582 & 0.000094 & 0.000676 & 0.000433 & 0.000507 & 0.001348 & 0.000675 & 0.000486 & 0.000426 \\ 0.000566 & 0.000204 & 0.000680 & 0.000451 & 0.000595 & 0.000675 & 0.001529 & 0.000479 & 0.000354 \\ 0.000476 & 0.000105 & 0.000791 & 0.000562 & 0.000376 & 0.000486 & 0.000479 & 0.001195 & 0.000298 \\ 0.000400 & 0.000077 & 0.000426 & 0.000275 & 0.000315 & 0.000426 & 0.000354 & 0.000298 & 0.000756 \end{matrix}$$

The classical model and the robust model, which have been defined in Section 2 and Section 3, are handled in the application. The models are given as follows:

The classical risk aversion portfolio optimization problem

$$\max\left(\mu' X - \lambda X' \Sigma X\right)$$

$$X' l = 1, \quad l = [1, 1, ..., 1]$$

The robust problem

$$max\left(\mu' X - \lambda X' \Sigma X - \delta\sqrt{X' \Sigma_\mu X}\right)$$

$$X' l = 1$$

In the first case, the problem (2.7) is solved for the different 20 values of λ, which is chosen by the information given in the Risk Aversion Formula by Fabozzi et al. [11]. The λ is chosen with an increase of 0.2. The [0,4] interval can be divided into smaller pieces so the number of λ can be increased.

For the different values of λ, the movement of the expected return, the variance and the objective function value are seen in the related Figure 1, Figure 2 and Figure 3.The figures show that when the value of λ increases (The aversion to risk is high–the risk-aversion investor), the values of the expected return, the variance and the objective function decrease. For the small values of λ (the aversion to risk

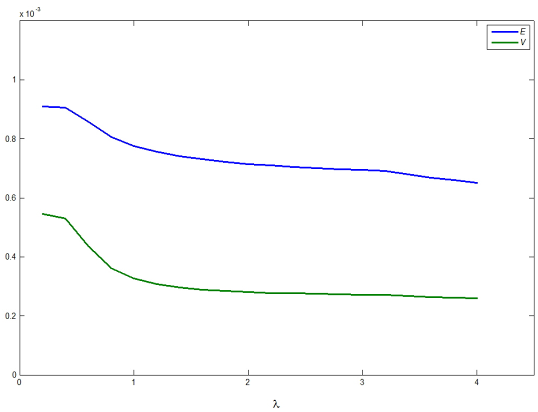

FIGURE 1. The movement of E : *Expected Value* and V : *Variance* for the classical problem

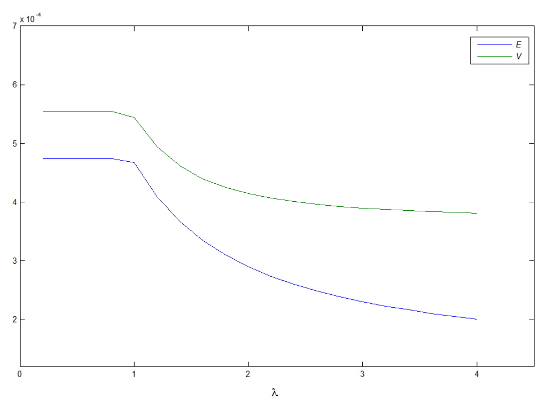

FIGURE 2. The movement of E and V for the robust problem

is low–the risk-lover investor), the values of the expected return, the variance and the objective function increase. In this case, it can be said that if the investor prefers the high expected return, the high risk must be considered.

In the robust case, when the expected return μ parameter is robust, the aim is to show the movement of the optimal solution to the parameter uncertainty. For this aim, the robust problem, which is given in (3.7), is solved for different λ and δ values.

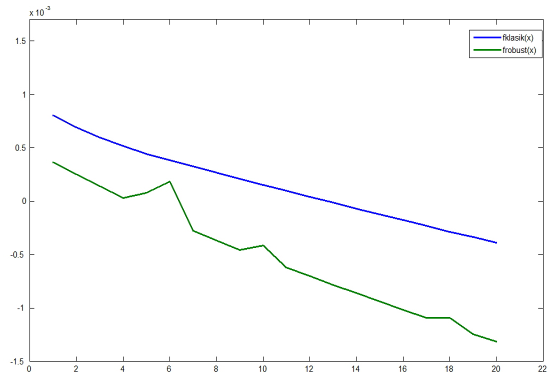

FIGURE 3. The movements of the objective functions for the classical problem and the robust problem

Table 1. The differences between the objective function values for the classical and the robust problem

$\boldsymbol{\lambda}$	$\mathbf{f\_classical(x) - f\_robust(x)}$
0.2	0.0004380
0.4	0.0004408
0.6	0.0004546
0.8	0.0004869
1.0	0.0003630
1.2	0.0002022
1.4	0.0006047
1.6	0.0006361
1.8	0.0006652
2.0	0.0005666
2.2	0.0007187
2.4	0.0007442
2.6	0.0007688
2.8	0.0007929
3.0	0.0008167
3.2	0.0008400
3.4	0.0008634
3.6	0.0008103
3.8	0.0009110
4.0	0.0009252

If there is a need to compare the solution of the robust problem with the solution of the classical problem, it is observed that for the same λ values, the expected return of the classical problem is larger than that of the robust problem. At the same time, its variance is smaller than the variance of the robust problem. In this situation, an investor should prefer the classical model.

On the other hand, the performances of portfolio models are measured by the Sharpe Ratio (SR) method. The Sharpe Ratio is defined as

$$SR = \frac{E(r_p)}{\sqrt{Var(r_p)}}.$$

High SR values mean high performance. The Sharpe Ratio values obtained for the classical and the robust problem are presented in Table 2.

Table 2. The Sharpe ratio values for the classical and the robust problem.

Sharpe ratio (Classical)	**Sharpe ratio (Robust)**
0.039004	0.020146
0.039332	0.020146
0.041081	0.020146
0.042440	0.020146
0.042931	0.020017
0.043053	0.018400
0.043017	0.017066
0.042916	0.015955
0.042789	0.015024
0.042655	0.014241
0.042526	0.013568
0.042404	0.012994
0.042285	0.012495
0.042081	0.012059
0.041943	0.011673
0.042081	0.011328
0.041514	0.011022
0.041113	0.010750
0.040747	0.010504
0.040390	0.010279

It is seen that the classical Sharpe Ratio values are higher than the robust Sharpe Ratio values. Therefore, it can be said that the performance of the classical problem is better than the performance of the robust problem. However, in the uncertainty situations, it is recommended that the robust problem be used, which works as well as the classical problem.

5. CONCLUSION

The portfolio optimization model of Harry Markowitz can be created in two frameworks, which minimize the risk of the portfolio for a certain level of expected return and maximize the return of the portfolio for a certain level of risk. In spite of the theoretical success of the mean-variance model, practitioners have avoided this model. The alternative model that combines the risk and the return of the objective function can be created using the coefficient of risk aversion. The solution of optimization problems is often very sensitive to perturbations in the parameters of the problem. In the literature, there are many alternative methods suggested to overcome the parameter perturbations. The robust optimization is one of the most commonly used models in the uncertainty case.

The results show that when the value of λ increases, the values of the expected return, the variance and the objective function decrease. It means that the aversion to risk is high here, so it can be said that the investor is the risk-aversion investor. For the small values of λ, on the other hand, the values of the expected return, the variance and the objective function increase. Here, the aversion to risk is low, so it means that the investor is the risk-lover investor. In this case, it can be said that if the investor prefers the high expected return, the high risk must be taken into consideration.

If the solution of the robust problem is to be compared with the solution of the classical problem, it is observed that for the same λ values, the expected return of the classical problem is larger than the robust problem. At the same time, its variance is smaller than the robust problem variance. In this situation, an investor should prefer the classical model. The classical solution obtained in the certainty situation and the solution obtained in the uncertainty situation give similar values at the objective function. Consequently, it can be said that the optimal solution in the uncertainty situation is robust to ambiguity of the parameter μ. The robust model, which works as well as the classical model in the uncertainty situations, can be used instead of the classical model.

Finally, the performances of portfolio models are measured by the Sharpe Ratio (SR) method. It is seen that the classical Sharpe Ratio values are higher than the robust Sharpe ratio values; however, the robust problem, which works well to overcome uncertainty, should be preferred in the uncertainty situations.

In the future studies, the problem can be solved for different values of $\lambda$and δ. The data was taken from the automotive sector in this study. In order to improve the study, it is possible to investigate other sectors. In the modelling part, the short-selling is forbidden. However, in theory it is possible to allow short-selling. Hence, allowing the sort-selling will be studied in the future. The solutions can be compared for all these situations.

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