



COEFFICIENT ESTIMATES FOR BI-CONCAVE FUNCTIONS

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ABSTRACT. In this study, a new class $\mathcal{C}_{\Sigma}^{p,q}(\alpha)$ of analytic and bi-concave functions were presented in the open unit disc. The coefficients estimates on the first two Taylor-Maclaurin coefficients $|a_2|$ and $|a_3|$ were found for functions belonging to this class.

1. INTRODUCTION, PRELIMINARIES AND DEFINITION

The knowledge on bi-concave univalent functions is based on univalent, concave and bi-univalent functions respectively. Therefore, a brief summary of these functions and related references are given in this section.

Lets take \mathbb{C} as the complex numbers and \mathbb{R} as the set of real numbers. Then open unit disk can be denoted by \mathbb{D} and extended complex plain are denoted by $\overline{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$. Let \mathcal{A} indicate the class of analytic functions in the open unit disk $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ given in the following form (1.1)

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n. \quad (1.1)$$

All the normalized analytic function classes \mathcal{A} which are univalent in \mathbb{D} are also represented by \mathcal{S} . An univalent function $f : \mathbb{D} \rightarrow \overline{\mathbb{C}}$ is called to be concave when $f(\mathbb{D})$ is concave, i.e. $\overline{\mathbb{C}} \setminus f(\mathbb{D})$ is convex.

Concave univalent functions have already been studied in detailed by several authors (see [1,2,3,4,7]).

A function $f : \mathbb{D} \rightarrow \mathbb{C}$ is called to be a member of concave univalent functions with an opening angle $\pi\alpha$, $\alpha \in (1, 2]$, at infinity if f holds the conditions given below:

- (i) f is analytic in \mathbb{D} which has normalization condition $f(0) = 0 = f'(0) - 1$. Additionally, f fulfills $f(1) = \infty$.

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(ii) f maps \mathbb{D} conformally onto a set whose complement in accordance with \mathbb{C} is convex.

(iii) The opening angle of $f(\mathbb{D})$ at infinity is equal to or less than $\pi\alpha$, $\alpha \in (1, 2]$.

Lets indicate the class of concave univalent functions of order β by $C_\beta(\alpha)$. The analytic characterization for functions in $C_\beta(\alpha)$ are as follows : For $\alpha \in (1, 2]$ and $\beta \in [0, 1)$, $f \in C_\beta(\alpha)$ if and only if

$$\operatorname{Re} P_f(z) > \beta, \quad \forall z \in \mathbb{D}, \quad (1.2)$$

for

$$P_f(z) = \frac{2}{\alpha - 1} \left[\frac{\alpha + 1}{2} \frac{1+z}{1-z} - 1 - \frac{zf''(z)}{f'(z)} \right] \quad \text{and} \quad f(0) = 0 = f'(0) - 1.$$

Especially, for $\beta = 0$, we can obtain the class of concave univalent functions $C_0(\alpha)$ which was studied in [3].

The closed set $\overline{\mathbb{C}} \setminus f(\mathbb{D})$ is convex and unbounded for $f \in C_0(\alpha)$, $\alpha \in (1, 2]$. $\forall f \in C_\beta(\alpha)$ has the Taylor expansion given by the following form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \quad |z| < 1.$$

For all $f \in \mathcal{S}$, the Koebe 1/4 theorem [8] confirms that the image of \mathbb{D} under all univalent function $f \in \mathcal{S}$ covers a disk of radius 1/4. Hence, each $f \in \mathcal{A}$ has f^{-1} , which is described by

$$f^{-1}(f(z)) = z \quad (z \in \mathbb{D})$$

and

$$f(f^{-1}(w)) = w \quad \left(|w| < r_0(f); r_0(f) \geq \frac{1}{4} \right).$$

If $f(z)$ is univalent in $|z| < 1$ and $g(w) = f^{-1}(w)$ is univalent in $|w| < 1$, then the function f belongs to analytic function is known to be bi-univalent in \mathbb{D} . If $f(z)$ given by (1.1) is bi-univalent, then $g = f^{-1}$ can be arranged in the form of Taylor expansion given Eqs. (1.3)

$$g(w) = w - a_2 w^2 + (2a_2^2 - a_3) w^3 - \dots \quad (1.3)$$

So, $f \in \mathcal{A}$ is called to be bi-univalent in \mathbb{D} if each of f and f^{-1} are univalent in \mathbb{D} . Also, a function f is bi-concave if both f and f^{-1} are concave.

Some properties of bi-convex, bi-univalent and bi-starlike function classes have already been investigated by Brannan and Taha [6]. Furthermore, an estimation of $|a_2|$ and $|a_3|$ was found by Bulut [5] for bi-starlike functions. Our results found for $|a_2|$ and $|a_3|$ are related to a different class, so called bi-concave functions.

Lets denote Σ as the class of all bi-univalent functions in the unit disk \mathbb{D} . Lewin [10] investigated Σ and showed that $|a_2| < 1.51$ for the function $f(z) \in \Sigma$. Also, several researchers obtained the coefficients boundary for $|a_2|$ and $|a_3|$ of bi-univalent functions for the some subclasses of the class Σ in references [9,11,12]. In addition, certain subclasses of bi-univalent functions, and also univalent functions consisting

of strongly starlike, starlike and convex functions were studied by Brannan and Taha [6]. They investigated bi-convex and bi-starlike functions and also investigated some properties of these classes.

Now, we define the definition of bi-concave functions as follows:

Definition 1.1. The function $f(z)$ in (1.1) is known to be $\sum_{C_\beta(\alpha)}$ if the conditions given below are fulfilled:

$$f \in \Sigma, \quad \Re \left\{ \frac{2}{\alpha - 1} \left[\frac{\alpha + 1}{2} \frac{1 + z}{1 - z} - 1 - \frac{zf''(z)}{f'(z)} \right] \right\} > \beta \quad , z \in \mathbb{D} \text{ and } 0 \leq \beta < 1 \quad (1.4)$$

and

$$\Re \left\{ \frac{2}{\alpha - 1} \left[\frac{\alpha + 1}{2} \frac{1 - w}{1 + w} - 1 - \frac{wg''(w)}{g'(w)} \right] \right\} > \beta \quad , w \in \mathbb{D} \text{ and } 0 \leq \beta < 1. \quad (1.5)$$

where the g is given in (1.3). In the other words, $\sum_{C_\beta(\alpha)}$ is the class of bi-concave functions order β .

The following definition is given by Xu et al. [13].

Definition 1.2. Lets define the functions $p, q : \mathbb{D} \rightarrow \mathbb{C}$ satisfying the following condition

$$\min \{ \Re(p(z)), \Re(q(z)) \} > 0 \quad (z \in \mathbb{D}) \text{ and } p(0) = q(0) = 1.$$

In addition let f , in (1.1), be in \mathcal{A} . Then, $f \in \mathcal{C}_\Sigma^{p,q}(\alpha)$ if the conditions given in (1.4) and (1.5) are fulfilled:

$$f \in \Sigma \quad \text{and} \quad \frac{2}{\alpha - 1} \left[\frac{\alpha + 1}{2} \frac{1 + z}{1 - z} - 1 - \frac{zf''(z)}{f'(z)} \right] \in p(\mathbb{D}), \quad (z \in \mathbb{D}) \quad (1.6)$$

and

$$\frac{2}{\alpha - 1} \left[\frac{\alpha + 1}{2} \frac{1 + w}{1 - w} - 1 - \frac{wg''(w)}{g'(w)} \right] \in q(\mathbb{D}), \quad (w \in \mathbb{D}) \quad (1.7)$$

where the g is given in (1.3).

Remark

If we let

$$p(z) = \frac{1 + (1 - 2\beta)z}{1 - z} \quad \text{and} \quad q(z) = \frac{1 - (1 - 2\beta)z}{1 + z} \quad (0 \leq \beta < 1, z \in \mathbb{D}) \quad (1.8)$$

in the class $\mathcal{C}_\Sigma^{p,q}(\alpha)$ then we have $\sum_{C_\beta(\alpha)}$.

The aim of this paper is to estimate the initial coefficients for the bi-concave functions in \mathbb{D} . The following theorem was proved for this purpose.

2. INITIAL COEFFICIENT BOUNDARY FOR $|a_2|$ AND $|a_3|$

The estimation of initial coefficient for bi-concave functions class $\mathcal{C}_{\Sigma}^{p,q}(\alpha)$ are presented in this section.

Theorem 2.1. *If the function $f(z)$ given by (1.1) is in $\mathcal{C}_{\Sigma}^{p,q}(\alpha)$ then*

$$|a_2| \leq \min \left\{ \sqrt{\frac{(\alpha+1)^2}{4} + \frac{(\alpha^2-1)}{8} [|p'(0)| + |q'(0)|] + \frac{(\alpha-1)^2}{32} [|p'^2 + |q'^2|]} \right. \\ \left. ; \sqrt{\frac{(\alpha+1)}{2} + \frac{(\alpha-1)}{16} [|p''(0)| + |q''(0)|]} \right\} \quad (2.1)$$

and

$$|a_3| \leq \min \left\{ \frac{(\alpha+1)}{2} + \frac{(\alpha-1)}{24} [2|p''(0)| + |q''(0)|] \right. \\ \left. ; \frac{(\alpha+1)^2}{4} + \frac{(\alpha-1)}{48} [|p''(0)| + |q''(0)|] + \frac{1}{8} (\alpha^2-1) [|p'(0)| + |q'(0)|] + \frac{1}{32} (\alpha-1)^2 [|p'^2 + |q'^2|] \right\}. \quad (2.2)$$

Proof. Firstly, we can write the argument inequalities in their equivalent forms as follows:

$$\frac{2}{\alpha-1} \left[\frac{(\alpha+1)}{2} \frac{1+z}{1-z} - 1 - \frac{zf''(z)}{f'(z)} \right] = p(z) \quad (z \in \mathbb{D}), \quad (2.3)$$

and

$$\frac{2}{\alpha-1} \left[\frac{(\alpha+1)}{2} \frac{1-w}{1+w} - 1 - \frac{wg''(w)}{g'(w)} \right] = q(w) \quad (w \in \mathbb{D}). \quad (2.4)$$

In addition to, the $p(z)$ and $q(w)$ can be expanded to Taylor-Maclaurin series as given below respectively

$$p(z) = 1 + p_1z + p_2z^2 + \dots$$

and

$$q(w) = 1 + q_1w + q_2w^2 + \dots$$

Now upon equating the coefficients of $\frac{2}{\alpha-1} \left[\frac{(\alpha+1)}{2} \frac{1+z}{1-z} - 1 - \frac{zf''(z)}{f'(z)} \right]$ with those of $p(z)$ and the coefficients of $\frac{2}{\alpha-1} \left[\frac{(\alpha+1)}{2} \frac{1-w}{1+w} - 1 - \frac{wg''(w)}{g'(w)} \right]$ with those of $q(w)$. We can write $p(z)$ and $q(w)$ as follows.

$$p(z) = \frac{2}{(\alpha-1)} \left[\frac{(\alpha+1)}{2} \frac{1+z}{1-z} - 1 - \frac{zf''(z)}{f'(z)} \right] = 1 + p_1z + p_2z^2 + p_3z^3 + \dots \quad (2.5)$$

and

$$q(w) = \frac{2}{(\alpha-1)} \left[\frac{(\alpha+1)}{2} \frac{1-w}{1+w} - 1 - \frac{wg''(w)}{g'(w)} \right] = 1 + q_1w + q_2w^2 + q_3w^3 + \dots \quad (2.6)$$

Since

$$\frac{zf''(z)}{f'(z)} = \frac{2a_2z + 6a_3z^2 + 12a_4z^3 + \dots}{1 + 2a_2z + 3a_3z^2 + 4a_4z^3 + \dots} = 2a_2z + (6a_3 - 4a_2^2)z^2 + \dots$$

and

$$\frac{1+z}{1-z} = 1 + 2 \sum_{n=1}^{\infty} z^n = 1 + 2z + 2z^2 + 2z^3 + \dots$$

we obtain that

$$\begin{aligned} & \frac{2}{\alpha-1} \left[\frac{(\alpha+1)}{2} \frac{1+z}{1-z} - 1 - \frac{zf''(z)}{f'(z)} \right] \\ = & \frac{2}{(\alpha-1)} \left[\frac{(\alpha+1)}{2} - 1 + (\alpha+1)z + (\alpha+1)z^2 + \dots - 2a_2z - (6a_3 - 4a_2^2)z^2 + \dots \right] \\ = & \frac{2}{(\alpha-1)} \left[\frac{(\alpha-1)}{2} + ((\alpha+1) - 2a_2)z + ((\alpha+1) - (6a_3 - 4a_2^2))z^2 + \dots \right] \\ = & 1 + \frac{2[(\alpha+1) - 2a_2]}{(\alpha-1)}z + \frac{2[(\alpha+1) - 6a_3 + 4a_2^2]}{(\alpha-1)}z^2 + \dots \end{aligned}$$

Then

$$p_1 = \frac{2[(\alpha+1) - 2a_2]}{(\alpha-1)} \quad (2.7)$$

$$p_2 = \frac{2[(\alpha+1) - 6a_3 + 4a_2^2]}{(\alpha-1)} \quad (2.8)$$

From (1.3) and (2.4)

$$\begin{aligned} \frac{wg''(w)}{g'(w)} &= \frac{-2a_2w + 6(2a_2^2 - a_3)w^2 - 12(5a_2^3 - 5a_2a_3 + a_4)w^3 + \dots}{1 - 2a_2w + 3(2a_2^2 - a_3)w^2 - 4(5a_2^3 - 5a_2a_3 + a_4)w^3 + \dots} \\ &= -2a_2w + (8a_2^2 - 6a_3)w^2. \end{aligned}$$

Then from $q(w)$ given by (2.6), we have

$$\begin{aligned} & \frac{2}{\alpha-1} \left[\frac{(\alpha+1)}{2} \frac{1-w}{1+w} - 1 - \frac{wg''(w)}{g'(w)} \right] \\ = & \frac{2}{(\alpha-1)} \left[\frac{(\alpha+1)}{2} - (\alpha+1)w + (\alpha+1)w^2 - \dots - 1 + 2a_2w - (8a_2^2 - 6a_3)w^2 + \dots \right] \\ = & 1 - \frac{2[(\alpha+1) - 2a_2]}{(\alpha-1)}w + \frac{2[(\alpha+1) - 8a_2^2 + 6a_3]}{(\alpha-1)}w^2 + \dots \end{aligned}$$

So we can obtain q_1 and q_2 as follows

$$q_1 = -\frac{2[(\alpha+1) - 2a_2]}{(\alpha-1)} \quad (2.9)$$

$$q_2 = \frac{2[(\alpha + 1) - 8a_2^2 + 6a_3]}{(\alpha - 1)}. \quad (2.10)$$

From (2.7) and (2.9) we obtain

$$p_1 = -q_1 \quad (2.11)$$

$$a_2^2 = \frac{(\alpha + 1)^2}{4} - \frac{(\alpha^2 - 1)}{8}[p_1 - q_1] + \frac{(\alpha - 1)^2}{32}[p_1^2 + q_1^2]. \quad (2.12)$$

Also, from (2.8) and (2.10) we obtain that

$$a_2^2 = \frac{(1 - \alpha)}{8}[p_2 + q_2] + \frac{4(\alpha + 1)}{8}. \quad (2.13)$$

Therefore, we find from the (2.11) and (2.12)

$$|a_2|^2 \leq \frac{(\alpha + 1)^2}{4} + \frac{(\alpha^2 - 1)}{8}[|p'(0)| + |q'(0)|] + \frac{(\alpha - 1)^2}{32}[|p'^2 + |q'^2]$$

and

$$|a_2|^2 \leq \frac{(\alpha + 1)}{2} + \frac{(\alpha - 1)}{16}[|p''(0)| + |q''(0)|].$$

So we have the coefficient of $|a_2|$ as maintained in (2.1).

Now, to obtain the bound on the coefficient $|a_3|$ we use (2.8) and (2.10). So we obtain

$$(\alpha - 1)(p_2 - q_2) = 24a_2^2 - 24a_3.$$

From (2.12) and (2.13) we find

$$\begin{aligned} 24a_3 &= -(\alpha - 1)(p_2 - q_2) + 24 \left(\frac{(\alpha + 1)}{2} + \frac{(1 - \alpha)}{8}(p_2 + q_2) \right) \\ &\Rightarrow a_3 = \frac{(\alpha + 1)}{2} - \frac{(\alpha - 1)}{12}[2p_2 + q_2]. \end{aligned} \quad (2.14)$$

We thus find that

$$|a_3| \leq \frac{\alpha + 1}{2} + \frac{(\alpha - 1)}{24}(2|p''(0)| + |q''(0)|).$$

Also from (2.12) and (2.13) we obtain

$$\begin{aligned} 24a_3 &= -(\alpha - 1)(p_2 - q_2) + 24 \left[\frac{(\alpha + 1)^2}{4} - \frac{(\alpha^2 - 1)}{8}(p_1 - q_1) + \frac{(\alpha - 1)^2}{32}(p_1^2 + q_1^2) \right] \\ &\Rightarrow a_3 = \frac{(\alpha + 1)^2}{4} - \frac{(\alpha - 1)}{24}(p_2 - q_2) - \frac{1}{8}(\alpha^2 - 1)(p_1 - q_1) + \frac{1}{32}(\alpha - 1)^2(p_1^2 + q_1^2). \end{aligned} \quad (2.15)$$

We thus find that

$$|a_3| \leq \frac{(\alpha + 1)^2}{4} + \frac{(\alpha - 1)}{48}(|p''(0)| + |q''(0)|) + \frac{1}{8}(\alpha^2 - 1)(|p'(0)| + |q'(0)|) + \frac{1}{32}(\alpha - 1)^2(|p'^2 + |q'^2).$$

So, The the proof of Theorem 2.1 is completed. \square

3. CONCLUSION

If p and q were chosen in Theorem 2.1 as follows, the following corollary can easily be obtained.

$$p(z) = \frac{1 + (1 - 2\beta)z}{1 - z} \quad \text{and} \quad q(z) = \frac{1 - (1 - 2\beta)z}{1 + z} \quad (0 \leq \beta < 1, z \in \mathbb{D})$$

Corollary 3.1. *Let $f(z)$, in the expansion (1.1) be in the bi-concave function class $\Sigma_{C_\beta(\alpha)}$, ($0 \leq \beta < 1$). Then*

$$|a_2| \leq \sqrt{\frac{(\alpha + 1)}{2} + \frac{(\alpha - 1)}{2}(1 - \beta)}$$

and

$$|a_3| \leq \frac{(\alpha + 1)}{2} + \frac{(\alpha - 1)}{2}(1 - \beta).$$

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