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COEFFICIENT ESTIMATES FOR BI-CONCAVE FUNCTIONS

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Abstract. In this study,a new class $\mathcal{C}_{\Sigma}^{p,q}(\alpha)$ of analytic and bi-concave functions were presented in the open unit disc. The coefficients estimates on the first two Taylor-Maclaurin coefficients $|a_2|$ and $|a_3|$ were found for functions belonging to this class.

1. Introduction, Preliminaries and Definition

The knowledge on bi-concave univalent functions is based on univalent, concave and bi-univalent funcions respectively. Therefore, a brief summary of these functions and related references are given in this section.

Lets take $\mathbb{C}$ as the complex numbers and $\mathbb{R}$ as the set of real numbers. Then open unit disk can be denoted by $\mathbb{D}$ and extended complex plain are denoted by $\overline{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$. Let $\mathcal{A}$ indicate the class of analytic functions in the open unit disk $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ given in the following form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n. \tag{1.1}$$

All the normalized analytic function classes $\mathcal{A}$ which are univalent in $\mathbb{D}$ are also represented by $\mathcal{S}$. An univalent function $f : \mathbb{D} \to \overline{\mathbb{C}}$ is called to be concave when $f(\mathbb{D})$ is concave, i.e. $\overline{\mathbb{C}} \setminus f(\mathbb{D})$ is convex.

Concave univalent functions have already been studied in detailed by several authors (see [1,2,3,4,7]).

A function $f : \mathbb{D} \to \mathbb{C}$ is called to be a member of concave univalent functions with an opening angle $\pi\alpha$, $\alpha \in (1, 2]$, at infinity if f holds the conditions given below:
(i) f is analytic in $\mathbb{D}$ which has normalization condition $f(0) = 0 = f'(0) - 1$. Additionaly, f fulfills $f(1) = \infty$.
(ii) f maps $\mathbb{D}$ conformally onto a set whose complement in accordance with $\mathbb{C}$ is

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convex.
(iii) The opening angle of $f(\mathbb{D})$ at infinity is equal to or less than $\pi\alpha$, $\alpha \in (1,2]$.

Lets indicate the class of concave univalent functions of order β by $C_\beta(\alpha)$.

The analytic characterization for functions in $C_\beta(\alpha)$ are as follows :
For $\alpha \in (1,2]$ and $\beta \in [0,1)$, $f \in C_\beta(\alpha)$ if and only if

$$\Re P_f(z) > \beta, \qquad \forall z \in \mathbb{D}, \tag{1.2}$$

for

$$P_f(z) = \frac{2}{\alpha-1}\left[\frac{\alpha+1}{2}\frac{1+z}{1-z} - 1 - \frac{zf''(z)}{f'(z)}\right] \quad and \quad f(0) = 0 = f'(0) - 1.$$

Especially, for $\beta = 0$, we can obtain the class of concave univalent functions $C_0(\alpha)$ which was studied in [3].

The closed set $\overline{\mathbb{C}} \setminus f(\mathbb{D})$ is convex and unbounded for $f \in C_0(\alpha)$, $\alpha \in (1,2]$. $\forall f \in C_\beta(\alpha)$ has the Taylor expansion given by the following form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \quad |z| < 1.$$

For all $f \in \mathcal{S}$, the Koebe 1/4 theorem [8] confirms that the image of $\mathbb{D}$ under all univalent function $f \in \mathcal{S}$ covers a disk of radius 1/4. Hence, each $f \in \mathcal{A}$ has f^{-1}, which is described by

$$f^{-1}(f(z)) = z \quad (z \in \mathbb{D})$$

and

$$f(f^{-1}(w)) = w \quad \left(|w| < r_0(f); r_0(f) \geq \frac{1}{4}\right).$$

If $f(z)$ is univalent in $\mathbb{D}$ and $g(w) = f^{-1}(w)$ is univalent in $\{w : \ |w| < 1\}$, then the function f belongs to analytic function is known to be bi-univalent in $\mathbb{D}$. If $f(z)$ given by (1.1) is bi-univalent, then $g = f^{-1}$ can be arranged in the form of Taylor expansion given

$$g(w) = w - a_2 w^2 + (2a_2^2 - a_3)w^3 - ... \quad . \tag{1.3}$$

So, $f \in \mathcal{A}$ is called to be bi-univalent in $\mathbb{D}$ if each of f and f^{-1} are univalent in $\mathbb{D}$. Also, a function f is bi-concave if both f and f^{-1} are concave.

Some properties of bi-convex, bi-univalent and bi-starlike function classes have already been investigated by Brannan and Taha [6]. Furthermore, an estimation of $|a_2|$ and $|a_3|$ was found by Bulut [5] for bi-starlike functions. Our results found for $|a_2|$ and $|a_3|$ are related to a different class, so called bi-concave functions.

Lets denote Σ as the class of all bi-univalent functions in the unit disk $\mathbb{D}$. Lewin [10] investigated Σ and showed that $|a_2| < 1.51$ for the function $f(z) \in \Sigma$. Also, several researchers obtained the coefficients boundary for $|a_2|$ and $|a_3|$ of bi-univalent functions for the some subclasses of the class Σ in references [9,11,12]. In addition, certain subclasses of bi-univalent functions, and also univalent functions consisting of strongly starlike, starlike and convex functions were studied by Brannan and Taha [6] . They investigated bi-convex and bi-starlike functions and also investigated some properties of these classes.

Now, we define the definition of bi-concave functions as follows:

Definition 1.1. The function $f(z)$ in (1.1) is known to be $\sum_{C_\beta(\alpha)}$, $(1<\alpha\leq 2)$ if the conditions given below are fulfilled: $f\in\Sigma$,

$$\Re\left\{\frac{2}{\alpha-1}\left[\frac{\alpha+1}{2}\frac{1+z}{1-z}-1-\frac{zf''(z)}{f'(z)}\right]\right\}>\beta \qquad ,z\in\mathbb{D}\ \mathit{and}\ 0\leq\beta<1 \tag{1.4}$$

and

$$\Re\left\{\frac{2}{\alpha-1}\left[\frac{\alpha+1}{2}\frac{1-w}{1+w}-1-\frac{wg''(w)}{g'(w)}\right]\right\}>\beta \qquad ,w\in\mathbb{D}\ \mathit{and}\ 0\leq\beta<1. \tag{1.5}$$

where the g is given in (1.3). In the other words, $\sum_{C_\beta(\alpha)}$ is the class of bi-concave functions order β.

We introduce the following subclass of the analytic functions class $\mathcal{A}$, analogously to the definition given by Xu et al. [13].

Definition 1.2. Lets define the functions $p,q:\mathbb{D}\to\mathbb{C}$ satisfying the following condition

$$\min\{\Re(p(z)),\Re(q(z))\}>0 \quad (z\in\mathbb{D})\ and\ p(0)=q(0)=1.$$

In addition let f, in (1.1), be in $\mathcal{A}$. Then, $f\in\mathcal{C}_\Sigma^{p,q}(\alpha)$, $(1<\alpha\leq 2)$ if the conditions given in (1.4) and (1.5) are fulfilled: $f\in\Sigma$

$$\frac{2}{\alpha-1}\left[\frac{\alpha+1}{2}\frac{1+z}{1-z}-1-\frac{zf''(z)}{f'(z)}\right]\in p(\mathbb{D}),\ (z\in\mathbb{D}) \tag{1.6}$$

and

$$\frac{2}{\alpha-1}\left[\frac{\alpha+1}{2}\frac{1-w}{1+w}-1-\frac{wg''(w)}{g'(w)}\right]\in q(\mathbb{D}),\ (w\in\mathbb{D}) \tag{1.7}$$

where the g is given in (1.3).

Remark

If we let

$$p(z)=\frac{1+(1-2\beta)z}{1-z} \qquad \mathit{and} \qquad q(z)=\frac{1-(1-2\beta)z}{1+z} \qquad (0\leq\beta<1,z\in\mathbb{D}) \tag{1.8}$$

in the class $\mathcal{C}_\Sigma^{p,q}(\alpha)$ then we have $\sum_{C_\beta(\alpha)}$.

The aim of this paper is to estimate the initial coefficients for the bi-concave functions in $\mathbb{D}$.

2. Initial Coefficient Boundary for $|a_2|$ and $|a_3|$

The estimation of initial coefficient for bi-concave functions class $\mathcal{C}_{\Sigma}^{p,q}(\alpha)$ are presented in this section.

Theorem 2.1. *If the function $f(z)$ given by (1.1) is in $\mathcal{C}_{\Sigma}^{p,q}(\alpha)$ then*

$$|a_2| \leq \min\left\{\sqrt{\frac{(\alpha+1)^2}{4} + \frac{(\alpha^2-1)}{8}[|p'(0)|+|q'(0)|] + \frac{(\alpha-1)^2}{32}[|p'^2+|q'^2]}\right. \\ \left. ;\sqrt{\frac{(\alpha+1)}{2} + \frac{(\alpha-1)}{16}[|p''(0)|+|q''(0)|]}\right\} \tag{2.1}$$

and

$$|a_3| \leq \min\left\{\frac{(\alpha+1)}{2} + \frac{(\alpha-1)}{24}[2|p''(0)|+|q''(0)|]\right. \\ \left. ;\frac{(\alpha+1)^2}{4} + \frac{(\alpha-1)}{48}[|p''(0)|+|q''(0)|] + \frac{1}{8}(\alpha^2-1)[|p'(0)|+|q'(0)|] + \frac{1}{32}(\alpha-1)^2[|p'^2+|q'^2]\right\}. \tag{2.2}$$

Proof. Firstly, we can write the argument inequalities in their equivalent forms as follows:

$$\frac{2}{\alpha-1}\left[\frac{(\alpha+1)}{2}\frac{1+z}{1-z} - 1 - \frac{zf''(z)}{f'(z)}\right] = p(z) \qquad (z\in\mathbb{D}), \tag{2.3}$$

and

$$\frac{2}{\alpha-1}\left[\frac{(\alpha+1)}{2}\frac{1-w}{1+w} - 1 - \frac{wg''(w)}{g'(w)}\right] = q(w) \qquad (w\in\mathbb{D}). \tag{2.4}$$

In addition to, the $p(z)$ and $q(w)$ can be expended to Taylor-Maclaurin series as given below respectively

$$p(z) = 1 + p_1z + p_2z^2 + ...$$

and

$$q(w) = 1 + q_1w + q_2w^2 + ... \quad .$$

Now upon equating the coefficients of $\frac{2}{\alpha-1}\left[\frac{(\alpha+1)}{2}\frac{1+z}{1-z} - 1 - \frac{zf''(z)}{f'(z)}\right]$ with those of $p(z)$ and the coefficients of $\frac{2}{\alpha-1}\left[\frac{(\alpha+1)}{2}\frac{1-w}{1+w} - 1 - \frac{wg''(w)}{g'(w)}\right]$ with those of $q(w)$. We can write $p(z)$ and $q(w)$ as follows.

$$p(z) = \frac{2}{(\alpha-1)}\left[\frac{(\alpha+1)}{2}\frac{1+z}{1-z} - 1 - \frac{zf''(z)}{f'(z)}\right] = 1 + p_1z + p_2z^2 + p_3z^3 + ... \tag{2.5}$$

and

$$q(w) = \frac{2}{(\alpha-1)}\left[\frac{(\alpha+1)}{2}\frac{1-w}{1+w} - 1 - \frac{wg''(w)}{g'(w)}\right] = 1 + q_1 w + q_2 w^2 + q_3 w^3 + \ldots \quad . \tag{2.6}$$

Since

$$\frac{zf''(z)}{f'(z)} = \frac{2a_2 z + 6a_3 z^2 + 12a_4 z^3 + \ldots}{1 + 2a_2 z + 3a_3 z^2 + 4a_4 z^3 + \ldots} = 2a_2 z + (6a_3 - 4a_2^2)z^2 + \ldots$$

and

$$\frac{1+z}{1-z} = 1 + 2\sum_{n=1}^{\infty} z^n = 1 + 2z + 2z^2 + 2z^3 + \ldots$$

we obtain that

$$\frac{2}{\alpha-1}\left[\frac{(\alpha+1)}{2}\frac{1+z}{1-z} - 1 - \frac{zf''(z)}{f'(z)}\right]$$

$$= \frac{2}{(\alpha-1)}\left[\frac{(\alpha+1)}{2} - 1 + (\alpha+1)z + (\alpha+1)z^2 + \ldots - 2a_2 z - (6a_3 - 4a_2^2)z^2 + \ldots\right]$$

$$= \frac{2}{(\alpha-1)}\left[\frac{(\alpha-1)}{2} + ((\alpha+1) - 2a_2)z + ((\alpha+1) - (6a_3 - 4a_2^2))z^2 + \ldots\right]$$

$$= 1 + \frac{2[(\alpha+1) - 2a_2]}{(\alpha-1)}z + \frac{2[(\alpha+1) - 6a_3 + 4a_2^2]}{(\alpha-1)}z^2 + \ldots \quad .$$

Then

$$p_1 = \frac{2[(\alpha+1) - 2a_2]}{(\alpha-1)} \tag{2.7}$$

$$p_2 = \frac{2[(\alpha+1) - 6a_3 + 4a_2^2]}{(\alpha-1)}. \tag{2.8}$$

From (1.3) and (2.4)

$$\frac{wg''(w)}{g'(w)} = \frac{-2a_2 w + 6(2a_2^2 - a_3)w^2 - 12(5a_2^3 - 5a_2 a_3 + a_4)w^3 + \ldots}{1 - 2a_2 w + 3(2a_2^2 - a_3)w^2 - 4(5a_2^3 - 5a_2 a_3 + a_4)w^3 + \ldots}$$

$$= -2a_2 w + (8a_2^2 - 6a_3)w^2 \cdots .$$

Then from $q(w)$ given by (2.6), we have

$$\frac{2}{\alpha-1}\left[\frac{(\alpha+1)}{2}\frac{1-w}{1+w} - 1 - \frac{wg''(w)}{g'(w)}\right]$$

$$= \frac{2}{(\alpha-1)}\left[\frac{(\alpha+1)}{2} - (\alpha+1)w + (\alpha+1)w^2 - \ldots - 1 + 2a_2 w - (8a_2^2 - 6a_3)w^2 + \ldots\right]$$

$$= 1 - \frac{2[(\alpha+1) - 2a_2]}{(\alpha-1)}w + \frac{2[(\alpha+1) - 8a_2^2 + 6a_3]}{(\alpha-1)}w^2 + \ldots \quad .$$

So we can obtain q_1 and q_2 as follows

$$q_1 = -\frac{2[(\alpha+1)-2a_2]}{(\alpha-1)} \tag{2.9}$$

$$q_2 = \frac{2[(\alpha+1)-8a_2^2+6a_3]}{(\alpha-1)} \quad . \tag{2.10}$$

From (2.7) and (2.9) we obtain

$$p_1 = -q_1 \tag{2.11}$$

$$a_2^2 = \frac{(\alpha+1)^2}{4} - \frac{(\alpha^2-1)}{8}[p_1-q_1] + \frac{(\alpha-1)^2}{32}[p_1^2+q_1^2]. \tag{2.12}$$

Also, from (2.8) and (2.10) we obtain that

$$a_2^2 = \frac{(1-\alpha)}{8}[p_2+q_2] + \frac{4(\alpha+1)}{8}. \tag{2.13}$$

Therefore, we find from the (2.12) and (2.13)

$$|a_2|^2 \leq \frac{(\alpha+1)^2}{4} + \frac{(\alpha^2-1)}{8}[|p'(0)|+|q'(0)|] + \frac{(\alpha-1)^2}{32}[|p'^2+|q'^2]$$

and

$$|a_2|^2 \leq \frac{(\alpha+1)}{2} + \frac{(\alpha-1)}{16}[|p''(0)|+|q''(0)|] \quad .$$

So we have the coefficient of $|a_2|$ as maintained in (2.1).
Now, to obtain the bound on the coefficient $|a_3|$ we use (2.8) and (2.10). So we obtain

$$(\alpha-1)(p_2-q_2) = 24a_2^2 - 24a_3.$$

From (2.13) we find

$$24a_3 = -(\alpha-1)(p_2-q_2) + 24\left(\frac{(\alpha+1)}{2} + \frac{(1-\alpha)}{8}(p_2+q_2)\right)$$

$$\Rightarrow a_3 = \frac{(\alpha+1)}{2} - \frac{(\alpha-1)}{12}[2p_2+q_2]. \tag{2.14}$$

We thus find that

$$|a_3| \leq \frac{\alpha+1}{2} + \frac{(\alpha-1)}{24}(2|p''(0)|+|q''(0)|).$$

Also from (2.12) we obtain

$$24a_3 = -(\alpha-1)(p_2-q_2) + 24\left[\frac{(\alpha+1)^2}{4} - \frac{(\alpha^2-1)}{8}(p_1-q_1) + \frac{(\alpha-1)^2}{32}(p_1^2+q_1^2)\right]$$

$$\Rightarrow a_3 = \frac{(\alpha+1)^2}{4} - \frac{(\alpha-1)}{24}(p_2-q_2) - \frac{1}{8}(\alpha^2-1)(p_1-q_1) + \frac{1}{32}(\alpha-1)^2(p_1^2+q_1^2). \tag{2.15}$$

We thus find that

$$|a_3| \leq \frac{(\alpha+1)^2}{4}+\frac{(\alpha-1)}{48}(|p''(0)|+|q''(0)|)+\frac{1}{8}(\alpha^2-1)(|p'(0)|+|q'(0)|)+\frac{1}{32}(\alpha-1)^2(|p'^2+|q'^2).$$

So, The the proof of Theorem 2.1 is completed. $\square$

3. Conclusion

If p and q are chosen in Theorem 2.1 as follows, the following corollary can easily be obtained.

$$p(z)=\frac{1+(1-2\beta)z}{1-z} \qquad \textit{and} \qquad q(z)=\frac{1-(1-2\beta)z}{1+z} \qquad (0\leq\beta<1, z\in\mathbb{D})$$

Corollary 3.1. *Let $f(z)$, in the expansion (1.1) be in the bi-concave function class $\sum_{C_\beta(\alpha)}$, $(0 \leq \beta < 1,\ \ 1<\alpha\leq 2)$. Then*

$$|a_2| \leq \sqrt{\frac{(\alpha+1)}{2}+\frac{(\alpha-1)}{2}(1-\beta)}$$

and

$$|a_3| \leq \frac{(\alpha+1)}{2}+\frac{(\alpha-1)}{2}(1-\beta).$$

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