



## GENERALIZATION OF $(\alpha - F_d)$ -CONTRACTION ON QUASI-METRIC SPACE

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**ABSTRACT.** In this paper, we introduce the concept of generalized  $(\alpha - F_d)$ -contraction and give some fixed point results in quasi metric spaces with different types of completeness.

### 1. INTRODUCTION

The beginning of metrical fixed point theory is related to the Banach's Contraction Principle, published in 1922. Banach's Contraction Principle says that, whenever  $(X, d)$  is complete, then any contraction self-map of  $X$  has a unique fixed point. This fixed point result is one of the most powerful tools for many existence and uniqueness problems arising in mathematics. Because of its importance, Banach Contraction Principle has been extended and generalized in many ways. Among all these, an interesting generalization was given by Wardowski [11] using a new concept  $F$ -contraction. Then many authors gave some results using this concept in different type metric space.

In this paper, we introduce the concept of generalized  $(\alpha - F_d)$ -contraction and give some fixed point results in quasi metric spaces with different types of completeness using this concept.

### 2. PRELIMINARIES

In this section, we give some basic definitions and properties about quasi metric space.

**Definition 1.** *Let  $X$  be any nonempty set. A function  $d : X \times X \rightarrow \mathbb{R}_+$  is said to be a quasi-pseudo metric if and only if for all  $x, y \in X$  the following conditions are satisfied:*

$$(i) \quad d(x, x) = 0,$$

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$$(ii) \quad d(x, y) \leq d(x, z) + d(z, y).$$

If a quasi-pseudo metric  $d$  satisfies:

$$\bullet \quad d(x, y) = d(y, z) = 0 \Rightarrow x = y$$

then  $d$  is said to be quasi metric.

Let  $d$  be a quasi metric. If  $d$  satisfies

$$d(x, y) = 0 \Rightarrow x = y$$

then  $d$  is said to be  $T_1$ -quasi metric.

It is clear that, every metric is a  $T_1$ -quasi metric, every  $T_1$ -quasi metric is a quasi metric and every quasi metric is a quasi-pseudo metric.

Let  $(X, d)$  be a quasi-pseudo metric space. Given a point  $x_0 \in X$  and a real constant  $\varepsilon > 0$ , the sets

$$B_d(x_0, \varepsilon) = \{y \in X : d(x_0, y) < \varepsilon\}$$

and

$$B_d[x_0, \varepsilon] = \{y \in X : d(x_0, y) \leq \varepsilon\}$$

are called open ball and closed ball with center  $x_0$  and radius  $\varepsilon$ .

Each quasi-pseudo metric  $d$  on  $X$  generates a topology  $\tau_d$  on  $X$  which has a base the family of open balls

$$\{B_d(x, \varepsilon) : x \in X \text{ and } \varepsilon > 0\}.$$

The closure and interior of a subset  $A$  of  $X$  with respect to  $\tau_d$  is denoted by  $cl_d(A)$  and  $int_d(A)$ , respectively.

If  $d$  is a quasi-pseudo metric on  $X$ , then  $\tau_d$  is a  $T_0$  topology and if  $d$  is a  $T_1$ -quasi-metric, then  $\tau_d$  is a  $T_1$  topology on  $X$ .

If  $d$  is a quasi metric and  $\tau_d$  is  $T_1$  topology, then  $d$  is  $T_1$ -quasi-metric.

If  $d$  is a quasi-pseudo metric on  $X$ , then the functions  $d^{-1}$ ,  $d^s$  and  $d_+$  defined by

$$d^{-1}(x, y) = d(y, x)$$

$$d^s(x, y) = \max \{d(x, y), d^{-1}(x, y)\}$$

and

$$d_+(x, y) = d(x, y) + d^{-1}(x, y)$$

are also quasi-pseudo metric on  $X$ . If  $d$  is a quasi metric, then  $d^s$  and  $d_+$  are (equivalent) metrics on  $X$ .

**Definition 2.** Let  $(X, d)$  be a quasi metric space and  $x \in X$ . The convergence of a sequence  $\{x_n\}$  to  $x$  with respect to  $\tau_d$  called  $d$ -convergence and denoted by  $x_n \xrightarrow{d} x$ , is defined

$$x_n \xrightarrow{d} x \Leftrightarrow d(x, x_n) \rightarrow 0.$$

Similarly the convergence of a sequence  $\{x_n\}$  to  $x$  with respect to  $\tau_{d^{-1}}$  called  $d^{-1}$ -convergence and denoted by  $x_n \xrightarrow{d^{-1}} x$ , is defined

$$x_n \xrightarrow{d^{-1}} x \Leftrightarrow d(x_n, x) \rightarrow 0.$$

Finally the convergence of a sequence  $\{x_n\}$  to  $x$  with respect to  $\tau_{d^s}$  called  $d^s$ -convergence and denoted by  $x_n \xrightarrow{d^s} x$ , is defined

$$x_n \xrightarrow{d^s} x \Leftrightarrow d^s(x_n, x) \rightarrow 0.$$

It is clear that  $x_n \xrightarrow{d^s} x \Leftrightarrow x_n \xrightarrow{d} x$  and  $x_n \xrightarrow{d^{-1}} x$ .

**Definition 3.** [8] Let  $(X, d)$  be a quasi metric space. A sequence  $\{x_n\}$  in  $X$  is called

- Left  $K$ -Cauchy (or forward Cauchy) if for every  $\varepsilon > 0$ , there exists  $n_0 \in \mathbb{N}$  such that

$$\forall n, k, n \geq k \geq n_0, d(x_k, x_n) < \varepsilon,$$

- Right  $K$ -Cauchy (or backward Cauchy) if for every  $\varepsilon > 0$ , there exists  $n_0 \in \mathbb{N}$  such that

$$\forall n, k, n \geq k \geq n_0, d(x_n, x_k) < \varepsilon,$$

- $d^s$ -Cauchy if for every  $\varepsilon > 0$ , there exists  $n_0 \in \mathbb{N}$  such that

$$\forall n, k \geq n_0, d(x_n, x_k) < \varepsilon.$$

It is clear that  $\{x_n\}$  is  $d^s$ -Cauchy if and only if it is both left  $K$ -Cauchy and right  $K$ -Cauchy. If a sequence is left  $K$ -Cauchy with respect to  $d$ , then it is right  $K$ -Cauchy with respect to  $d^{-1}$ . If  $\{x_n\}$  is a sequence in a quasi metric space  $(X, d)$  such that

$$\sum_{n=1}^{\infty} d(x_n, x_{n+1}) < \infty,$$

then it is left  $K$ -Cauchy sequence.

It is well known that every convergent sequence is Cauchy sequence in a metric space. In general this situation is not valid in a quasi metric space. That is,  $d$ -convergent or  $d^{-1}$ -convergent sequences may not be Cauchy (in the sense of  $d^s$ , left  $K$  and right  $K$ ) in a quasi metric space.

**Definition 4.** Let  $(X, d)$  be a quasi metric space. Then  $(X, d)$  is said to be

- bicomplete if every  $d^s$ -Cauchy sequence is  $d^s$ -convergent,
- left (right)  $K$ -complete if every left (right)  $K$ -Cauchy sequence is  $d$ -convergent,
- left (right)  $\mathcal{M}$ -complete if every left (right)  $K$ -Cauchy sequence is  $d^{-1}$ -convergent,
- left (right) Smyth complete if every left (right)  $K$ -Cauchy sequence is  $d^s$ -convergent.

One can find more information about important properties of quasi metric space in [8, 9].

On the other hand,  $\alpha$ -admissibility and  $F$ -contractivity of a mapping are popular in recent years. The concept of  $\alpha$ -admissibility of a mapping on a nonempty set has been introduced by Samet et. al. [10]. Let  $X$  be a nonempty set,  $T$  be a self mapping of  $X$  and  $\alpha : X \times X \rightarrow [0, \infty)$  be a function. Then  $T$  is said to be  $\alpha$ -admissible if

$$\alpha(x, y) \geq 1 \Rightarrow \alpha(Tx, Ty) \geq 1.$$

Using this concept, Samet et al. [10] provided some general fixed point results including many known theorems on complete metric spaces.

In 2012, Wardowski [11] introduced the concept of  $F$ -contraction. Let  $\mathcal{F}$  be the family of all functions  $F : (0, \infty) \rightarrow \mathbb{R}$  satisfying the following conditions:

- (F1)  $F$  is strictly increasing, i.e., for all  $\alpha, \beta \in (0, \infty)$  such that  $\alpha < \beta$ ,  $F(\alpha) < F(\beta)$ ,
- (F2) for each sequence  $\{a_n\}$  of positive numbers

$$\lim_{n \rightarrow \infty} a_n = 0 \text{ if and only if } \lim_{n \rightarrow \infty} F(a_n) = -\infty,$$

- (F3) there exists  $k \in (0, 1)$  such that  $\lim_{\alpha \rightarrow 0^+} \alpha^k F(\alpha) = 0$ .

**Definition 5.** Let  $(X, d)$  be a metric space and  $T : X \rightarrow X$  be a mapping. Then  $T$  is said to be an  $F$ -contraction if  $F \in \mathcal{F}$  and there exists  $\tau > 0$  such that

$$\forall x, y \in X [d(Tx, Ty) > 0 \Rightarrow \tau + F(d(Tx, Ty)) \leq F(d(x, y))].$$

Various fixed point results for  $\alpha$ -admissible mappings and  $F$ -contractions on complete metric space can be found in [1, 3, 4, 5] and [1, 2, 3, 7, 12], respectively.

### 3. MAIN RESULTS

Let  $(X, d)$  be a quasi metric space,  $T : X \rightarrow X$  be a mapping and  $\alpha : X \times X \rightarrow [0, \infty)$  be a function. We will consider the following set (see [1]):

$$T_\alpha = \{(x, y) \in X \times X : \alpha(x, y) \geq 1 \text{ and } d(Tx, Ty) > 0\}. \quad (3.1)$$

**Definition 6.** Let  $(X, d)$  be a quasi metric space and  $T : X \rightarrow X$  be a mapping satisfying

$$d(x, y) = 0 \Rightarrow d(Tx, Ty) = 0. \quad (3.2)$$

$\alpha : X \times X \rightarrow [0, \infty)$  and  $F \in \mathcal{F}$  be functions. Then  $T$  is said to be a generalized  $(\alpha - F_d)$ -contraction if there exists  $\tau > 0$  such that

$$\tau + F(d(Tx, Ty)) \leq F(M(x, y)) \quad (3.3)$$

for all  $x, y \in T_\alpha$  where

$$M(x, y) = \max \left\{ d(x, y), d(x, Tx), d(y, Ty), \frac{1}{2} [d(x, Ty) + d(y, Tx)] \right\}.$$

**Theorem 1.** *Let  $(X, d)$  be a Hausdorff left  $K$ -complete quasi metric space,  $T : X \rightarrow X$  be a generalized  $(\alpha - F_d)$ -contraction. Assume that  $T$  is  $\alpha$ -admissible and  $\tau_d$ -continuous. If there exists  $x_0 \in X$  such that  $\alpha(x_0, Tx_0) \geq 1$ , then  $T$  has a fixed point in  $X$ .*

*Proof.* Let  $x_0 \in X$  be such that  $\alpha(x_0, Tx_0) \geq 1$ . Define a sequence  $\{x_n\}$  in  $X$  by  $x_n = Tx_{n-1}$  for all  $n \in \mathbb{N}$ . Since  $T$  is  $\alpha$ -admissible, then  $\alpha(x_n, x_{n+1}) \geq 1$  for all  $n \in \mathbb{N}$ . Now let

$$d_n = d(x_n, x_{n+1}) \quad (3.4)$$

for all  $n \in \mathbb{N}$ . If there exists  $k \in \mathbb{N}$  with  $d_k = d(x_k, x_{k+1}) = 0$  then  $x_k$  is a fixed point of  $T$  since  $d$  is  $T_1$  quasi metric space. Suppose  $d_n > 0$  for all  $n \in \mathbb{N}$ . Since  $T$  is generalized  $(\alpha - F_d)$ -contraction, we get

$$\tau + F(d(x_n, x_{n+1})) \leq F(M(x_{n-1}, x_n)) \quad (3.5)$$

where

$$\begin{aligned} & M(x_{n-1}, x_n) \\ = & \max \left\{ d(x_{n-1}, x_n), d(x_{n-1}, x_n), d(x_n, x_{n+1}), \frac{1}{2} [d(x_n, x_{n+2}) + d(x_{n+1}, x_{n+1})] \right\} \\ = & \max \{d(x_{n-1}, x_n), d(x_n, x_{n+1})\}. \end{aligned}$$

So, we have

$$\begin{aligned} \tau + F(d(x_n, x_{n+1})) & \leq F(\max \{d(x_{n-1}, x_n), d(x_n, x_{n+1})\}) \\ & \leq F(\max \{d(x_{n-1}, x_n), d(x_n, x_{n+1})\}). \end{aligned}$$

If  $\max \{d(x_{n-1}, x_n), d(x_n, x_{n+1})\} = d(x_n, x_{n+1})$ , then

$$\tau + F(d(x_n, x_{n+1})) \leq F(d(x_n, x_{n+1}))$$

which is a contradiction. Hence,  $\max \{d(x_{n-1}, x_n), d(x_n, x_{n+1})\} = d(x_{n-1}, x_n)$ , then we obtain

$$\tau + F(d_n) \leq F(d_{n-1}).$$

Continuing this process, we get

$$F(d_n) \leq F(d_{n-1}) - \tau \leq F(d_{n-2}) - 2\tau \leq \dots \leq F(d_0) - n\tau. \quad (3.6)$$

From 3.6, we get  $\lim_{n \rightarrow \infty} F(d_n) = -\infty$ . Thus from (F2), we get

$$\lim_{n \rightarrow \infty} d_n = 0.$$

From (F3) there exists  $k \in (0, 1)$  such that

$$\lim_{n \rightarrow \infty} d_n^k F(d_n) = 0.$$

By 3.6, the following holds for all  $n \in \mathbb{N}$

$$d_n^k F(d_n) - d_n^k F(d_0) \leq -d_n^k n\tau \leq 0. \quad (3.7)$$

Letting  $n \rightarrow \infty$  in 3.7, we obtain that

$$\lim_{n \rightarrow \infty} nd_n^k = 0. \quad (3.8)$$

From 3.8, there exists  $n_1 \in \mathbb{N}$  such that  $nd_n^k \leq 1$  for all  $n \geq n_1$ . So we have for all  $n \geq n_1$

$$d_n \leq \frac{1}{n^{\frac{1}{k}}}. \quad (3.9)$$

Therefore  $\sum_{n=1}^{\infty} d_n < \infty$ . Now, let  $m, n \in \mathbb{N}$  with  $m > n \geq n_1$ , then we get

$$\begin{aligned} d(x_n, x_m) &\leq d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+2}) + \cdots + d(x_{m-1}, x_m) \\ &= d_n + d_{n+1} + \cdots + d_{m-1} \\ &\leq \sum_{k=n}^{\infty} d_k. \end{aligned}$$

Since  $\sum_{k=1}^{\infty} d_k$  is convergent, then we get  $\{x_n\}$  is left  $K$ -Cauchy sequence in the quasi metric space  $(X, d)$ . Since  $(X, d)$  left  $K$ -complete, there exists  $z \in X$  such that  $\{x_n\}$  is  $d$ -converges to  $z$ , that is,  $d(z, x_n) \rightarrow 0$  as  $n \rightarrow \infty$ . Since  $T$  is  $\tau_d$ -continuous, then

$$d(Tz, Tx_n) = d(Tz, x_{n+1}) \rightarrow 0 \text{ as } n \rightarrow \infty.$$

Since  $X$  is Hausdorff, we get  $z = Tz$ .  $\square$

**Theorem 2.** *Let  $(X, d)$  be a Hausdorff left  $\mathcal{M}$ -complete quasi metric space,  $T : X \rightarrow X$  be a generalized  $(\alpha - F_d)$ -contraction. Assume that  $T$  is  $\alpha$ -admissible and  $\tau_{d^{-1}}$ -continuous. If there exists  $x_0 \in X$  such that  $\alpha(x_0, Tx_0) \geq 1$ , then  $T$  has a fixed point in  $X$ .*

*Proof.* As in the proof of Theorem 1, we can get the iterative sequence  $\{x_n\}$  is left  $K$ -Cauchy. Since  $(X, d)$  is left  $\mathcal{M}$ -complete, there exists  $z \in X$  such that  $\{x_n\}$  is  $d^{-1}$ -converges to  $z$ , that is,  $d(x_n, z) \rightarrow 0$  as  $n \rightarrow \infty$ . By  $\tau_{d^{-1}}$ -continuity of  $T$ , we have

$$d(Tx_n, Tz) \rightarrow 0 \text{ as } n \rightarrow \infty.$$

Since  $X$  is Hausdorff, we get  $z = Tz$ .  $\square$

**Theorem 3.** *Let  $(X, d)$  be left Smyth complete  $T_1$ -quasi metric space,  $T : X \rightarrow X$  be a generalized  $(\alpha - F_d)$ -contraction. Assume that  $T$  is  $\alpha$ -admissible and  $\tau_d$  or  $\tau_{d^{-1}}$ -continuous. If there exists  $x_0 \in X$  such that  $\alpha(x_0, Tx_0) \geq 1$ , then  $T$  has a fixed point in  $X$ .*

*Proof.* As in the proof of Theorem 1, we can get the iterative sequence  $\{x_n\}$  is left  $K$ -Cauchy. Since  $(X, d)$  is left Smyth complete, there exists  $z \in X$  such that  $\{x_n\}$  is  $d^s$ -converges to  $z$ , that is,  $d^s(x_n, z) \rightarrow 0$  as  $n \rightarrow \infty$ . If  $T$  is  $\tau_d$ -continuous, then

$$d(Tz, Tx_n) = d(Tz, x_{n+1}) \rightarrow 0 \text{ as } n \rightarrow \infty.$$

Therefore we get

$$d(Tz, z) \leq d(Tz, x_{n+1}) + d(x_{n+1}, z) \rightarrow 0 \text{ as } n \rightarrow \infty.$$

If  $T$  is  $\tau_{d-1}$ -continuous, then

$$d(Tx_n, Tz) = d(x_{n+1}, Tz) \rightarrow 0 \text{ as } n \rightarrow \infty.$$

Therefore we get

$$d(z, Tz) \leq d(z, x_{n+1}) + d(x_{n+1}, Tz) \rightarrow 0 \text{ as } n \rightarrow \infty.$$

Since  $T$  is  $T_1$ -quasi metric space, we obtain  $z = Tz$ . □

**Example 1.** Let  $X = \{0, 2, 4, \dots\}$  and

$$d(x, y) = \begin{cases} 0 & ; x = y \\ x + y & ; x \neq y \end{cases}$$

for all  $x, y \in X$ . Now define  $T : X \rightarrow X$  as

$$Tx = \begin{cases} 0 & ; x = 0 \\ x - 2 & ; x \neq 0 \end{cases}$$

and  $\alpha : X \times X \rightarrow [0, \infty)$  as

$$\alpha(x, y) = \begin{cases} 0; & (x, y) \in \{(0, 2), (2, 0)\} \\ 2; & \text{otherwise} \end{cases}$$

Since  $(X, d)$  is complete metric space then it is left Smyth complete  $T_1$ -quasi metric space. Also  $T$  is  $\alpha$ -admissible. On the other hand

$$\begin{aligned} T_\alpha &= \{(x, y) \in X \times X : \alpha(x, y) \geq 1 \text{ and } d(Tx, Ty) > 0\} \\ &= \{(x, y) \in X \times X : (x, y) \notin \{(0, 2), (2, 0)\} \text{ and } x \neq y\} \end{aligned}$$

Then we get  $T$  is a generalized  $(\alpha - F_d)$ -contraction with  $F(\delta) = \ln \delta$ ,  $\tau = \ln 2$  for all  $x, y \in T_\alpha$  and  $z = 0$  is a fixed point of  $T$ .

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