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MAXIMUM LQ-LIKELIHOOD ESTIMATION FOR THE PARAMETERS OF MARSHALL-OLKIN EXTENDED BURR XII DISTRIBUTION

ŞENAY ÖZDEMIR, YEŞIM GÜNEY, YETKIN TUAÇ, AND OLCAY ARSLAN

Abstract. Marshall–Olkin extended Burr XII (MOEBXII) distribution is proposed by Al-Saiari et al. (2014) to obtain a more flexible family of distributions. Some estimation methods like maximum likelihood, Bayes and M estimations are used to estimate the parameters of the MOEBXII distribution in literature. In this paper, we propose to use Maximum Lq (MLq) estimation method to find alternative estimators for the parameters of the MOEBXII distribution. We give some simulation studies and a real data example to compare the performance of the MLq estimators with the maximum likelihood and M estimators. According to our results MLq estimation method is a good alternative to the maximum likelihood and M estimation methods in the presence of outliers.

1. Introduction

The Burr XII distribution is a member of a system which contains twelve distributions defined by Burr [3]. The distribution has relationship with various distributions like generalized Beta II, some mixtures of Weibull, logistic and Lomax. Therefore, it can be used to identify data sets from a wide variety fields such as financial [20], actuarial sciences [18], reliability and survival analysis [1, 13, 21, 22].

The Burr XII distribution has the probability density function (pdf)

$$f(x;c,k) = ck\frac{x^{(c-1)}}{(1+x^c)^{k+1}},\ x \geq 0, c > 0, k > 0 \tag{1.1}$$

where $c > 0$ and $k > 0$ both are the shape parameters. The cumulative density function (cdf) of Burr XII distribution is as follows

$$F(x;c,k) = 1 - \frac{1}{(1+x^c)^k}\ , x \geq 0. \tag{1.2}$$

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It is useful to generalize a distribution to provide flexibility and to model the data sets with various shaped. For this reason a number of extension methods have been proposed in literature. One of the most popular methods is to use Marshall Olkin (MO) transformation [19]. The MO transformation provides an extra parameter which yields a better fit with different types of data than baseline distribution. MO extended distributions and their applications have been extensively studied in the literature such as [4, 9, 10, 11, 12, 14, 24]. The MO transformation is defined as given in below.

Let $\overline{F}(x) = 1 - F(x)$ is the survival function of the baseline distribution. Then the Marshall-Olkin (MO) extended distribution can be defined with the following survival function

$$\overline{F}(x) = \frac{\alpha \overline{F}(x)}{1 - \overline{\alpha}\overline{F}(x)}, -\infty < x < \infty, \alpha > 0 \tag{1.3}$$

where $\overline{\alpha} = 1 - \alpha$. It can be easily seen that the MO extended distribution reduces the baseline distribution when $\alpha = 1$. Further the pdf of MO extended distribution is given by

$$f(x) = \frac{\alpha f(x)}{\left[1 - \overline{\alpha}\overline{F}(x)\right]^2}. \tag{1.4}$$

By considering the Burr XII distribution as the baseline distribution, Marshall Olkin Extended Burr XII (MOEBXII) distribution have been proposed by [2]. Combining the equation (1.1) and equation (1.4), the pdf of the $MOEBXII(\alpha, c, k)$ distribution is obtained as follows

$$f(x; \alpha, c, k) = \alpha ck \frac{x^{(c-1)}(1 + x^c)^{-(k+1)}}{\left[1 - (1 - \alpha)(1 + x^c)^{-k}\right]^2}, \ x \geq 0, \ \alpha, c, k > 0. \tag{1.5}$$

The cdf of this proposed distribution is given by

$$F(x; \alpha, c, k) = \frac{1 - (1 + x^c)^{-k}}{1 - (1 - \alpha)(1 + x^c)^{-k}}$$

where $x \geq 0$ and $\alpha, c, k > 0$.

The parameters of the MOEBXII can be estimated by using classical methods such as maximum likelihood (ML) method and Bayesian method [2]. However, these estimation methods are sensitive to the outliers in the data. Therefore, Güney and Arslan [15] have proposed a robust estimation method based on M-estimation for the parameters of the MOEBXII. Another estimation method which decreases the influence of the outliers is the Maximum Lq-likelihood (MLq) estimation based on q-order entropy. The q order entropy, which is proposed by Havrda and Charvat [16], has the function

$$Lq(u) = \begin{cases} \log u & , \quad q = 1 \\ (u^{1-q} - 1)/(1 - q) & , \quad otherwise \end{cases}.$$

in place of $\log u$ in Shannon's entropy, where u is the pdf. Maximizing likelihood function is equivalent to minimize negative log likelihood function which relies on Shannon entropy. To estimate the parameters, MLq likelihood estimation, which is proposed by Ferrari et al.[7], use the $Lq(u)$ function instead of loglikelihood function as in the ML estimation method. The MLq estimation method has a distortion parameter q which has an effect on the role of observations. The MLq estimation method includes $(1-q)$th power of density as a weight, therefore it performs as a robust estimation method. If $q=1$, all the observations have the same weights as equal to the ML method. Namely, the MLq estimator approaches the ML estimator when $q \to 1$ [7]. The MLq method reduces to the effect of the extreme observations on the parameter estimations by the help of the distortion parameter q. Choosing q is an another challenging problem in the MLq estimation. In this study, we take $q = 1 - \frac{1}{n}$ as given by [7] for sake of simplicity.

Consider that $\underline{x} = (x_1, x_2, ..., x_n)$ be an independent identically distribution (i.i.d.) sample from the pdf $f(x;\theta)$ with $\theta \in \Theta \subseteq \mathbb{R}^p$. The MLq estimation of θ is

$$\widehat{\theta}_n = \arg\max_{\theta \epsilon \Theta} \sum_{i=1}^{n} Lq[f(x_i, \theta)], q > 0 \tag{1.6}$$

where

$$Lq(u) = \begin{cases} \log u & , \quad q = 1 \\ (u^{1-q} - 1)/(1-q) & , \quad otherwise \end{cases} .$$

After Ferrari et al. [7] introduced the MLq estimation, some studies have been carried out about MLq estimation method such as [8, 23, 25] Some of this studies give information about efficiency of MLq estimation while [25] emphasize its robustness. In this paper we studied on the MLq estimator for the parameters of the MOEBXII and compare them with some estimators by means of mean square error (MSE) values.

The rest of the paper is organized as follows: In Section 2, ML, robust M and MLq estimation methods are summarized to obtain the estimates of the parameters of $MOEBXII(\alpha, c, k)$. Some simulation studies are conducted in Section 3 to compare the performance of the parameter estimation methods that we consider in this paper. A CO_2 emission data from different countries is investigated in Section 4. Finally we conclude the paper with a conclusion in Section 5.

2. Estimation of the Parameters of MOEBXII Distribution

In this section we give the ML, M and the MLq estimation of the parameters of MOEBXII distribution.

2.1. Maximum Likelihood Estimation of MOEBXII Distribution.

Suppose that $\underline{x} = (x_1, x_2, ..., x_n)$ be an i.i.d. random sample from the $MOEBXII(\alpha, c, k)$. As known as the ML estimators can be obtained by maximizing the log-likelihood function with respect to the parameters of interest. Log-likelihood function of the $MOEBXII(\alpha, c, k)$ is

$$\begin{aligned} l(\alpha, c, k) &= n\log(\alpha ck) + (c-1)\sum_{i=1}^{n}\log x_i - (k+1)\sum_{i=1}^{n}\log(1+x_i^c) \\ &\quad -2\sum_{i=1}^{n}\log\left(1-(1-\alpha)(1+x_i^c)^{-k}\right). \end{aligned} \tag{2.1}$$

The derivatives of $l(\alpha, c, k)$ with respect to α, c and k can be written as

$$\frac{\partial l}{\partial \alpha} = \frac{n}{\alpha} - 2\sum_{i=1}^{n}\frac{(1+x_i^c)^{-k}}{1-(1-\alpha)(1+x_i^c)^{-k}} = 0, \tag{2.2}$$

$$\begin{aligned} \frac{\partial l}{\partial c} &= \frac{n}{c} + \sum_{i=1}^{n}\log x_i - (k+1)\sum_{i=1}^{n}\frac{x_i^c\log(x_i)}{(1+x_i^c)} \\ &\quad -2k(1-\alpha)\sum_{i=1}^{n}\frac{x_i^c(1+x_i^c)^{-(k+1)}\log(x_i)}{1-(1-\alpha)(1+x_i^c)^{-k}} = 0, \end{aligned} \tag{2.3}$$

and

$$\frac{\partial l}{\partial k} = \frac{n}{k} - \sum_{i=1}^{n}\log(1+x_i^c) - 2(1-\alpha)\sum_{i=1}^{n}\frac{(1+x_i^c)^{-k}\log(1+x_i^c)}{1-(1-\alpha)(1+x_i^c)^{-k}} = 0. \tag{2.4}$$

Since there are no explicit solutions for (2.2), (2.3) and (2.4), numerical methods are used to solve this equation system.

2.2. **M Estimation for the MOEBXII Distribution.** To deal with outliers in data M estimation method can be used to obtain robust estimators for the parameters of Burr XII [6] and MOEBXII distribution. The M estimation method estimate the parameters of interest by minimizing the following objective function with the Huber or Tukey ρ function [17]. M estimation method is considered by Güney and Arslan [15] for the MOEBXII distribution.

$$Q(\alpha, c, k) = \sum_{i=1}^{n}\rho\left(y_i - \log\log(\alpha-1) + \log\log(\alpha) - \log(k) - \log\log(1+x_i^c)\right). \tag{2.5}$$

Taking the derivatives of the objective function Q with respect to each parameter and adjusting the equations, M estimators of MOEBXII parameters can be obtained by solving the following equations:

$$\log \widehat{k} = \frac{\sum_{i=1}^{n} \omega_i y_i}{\sum_{i=1}^{n} \omega_i} - \frac{\sum_{i=1}^{n} \omega_i \log\log(1+x_i^c)}{\sum_{i=1}^{n} \omega_i} - \log\log(\alpha-1) + \log\log(\alpha), \tag{2.6}$$

$$\sum_{i=1}^{n} \left(\begin{array}{c} \omega_i \left(y_i - \log\log(\alpha-1) + \log\log(\alpha) - \log(k) - \log\log(1+x_i^c)\right) \\ \times \frac{x_i^c \log(x_i)}{(1+x_i^c)\log(1+x_i^c)} \end{array} \right) = 0, \tag{2.7}$$

$$\log\left(\log(\alpha)\log(\alpha-1)\right) = \frac{\sum_{i=1}^{n} \omega_i y_i}{\sum_{i=1}^{n} \omega_i} - \frac{\sum_{i=1}^{n} \omega_i \log\log(1+x_i^c)}{\sum_{i=1}^{n} \omega_i} - \log(k) \tag{2.8}$$

where

$$\omega_i = \min\left\{1, \frac{b_1}{\left|\left(y_i - \log\log(\alpha-1) + \log\log(\alpha) - \log(k) - \log\log(1+x_i^c)\right)\right|}\right\} \tag{2.9}$$

and

$$\omega_i = \left(\begin{array}{c} \left(1 - \left(\frac{(y_i - \log\log(\alpha-1) + \log\log(\alpha) - \log(k) - \log\log(1+x_i^c))}{b_2}\right)^2\right)^2 \\ \times I\left(\left|\left(y_i - \log\log(\alpha-1) + \log\log(\alpha) - \log(k) - \log\log(1+x_i^c)\right)\right| \le b_2\right) \end{array} \right) \tag{2.10}$$

for the Huber's ρ function and Tukey's ρ function, respectively [15], since Huber's ρ function is

$$\rho(x) = \begin{cases} x^2 & , \quad |x| \le b_1 \\ 2b_1 |x| - b_1^2 & , \quad |x| > b_1 \end{cases}$$

and Tukey's ρ functions is

$$\rho(x) = \begin{cases} 1 - \left(1 - (x/b_2)^2\right)^2 & , \quad |x| \le b_2 \\ 1 & , \quad |x| > b_2 \end{cases}$$

where b_1 and b_2 are robustness tuning constants.

M estimates of interested parameters can be found by solving these equations numerically because there are no analytical solutions for them.

2.3. **Maximum Lq Likelihood Estimation of the MOEBXII Distribution.** MLq estimation for the parameters of a distribution with the pdf $f(x;\theta)$ is given in equation (1.6). To obtain the MLq estimators define

$$U(x,\theta) = \nabla_\theta \left[\log f(x;\theta)\right],$$
$$U^*(x,\theta,q) = U(x,\theta) f(x;\theta)^{1-q}.$$

Then the solution of estimating equation

$$\sum_{i=1}^{n} U^*(x,\theta,q) = 0$$

gives the MLq estimator $\widehat{\theta}_n$ [7].

Let $\underline{x} = (x_1, x_2, ..., x_n)$ be an i.i.d. sample of size n from the MOEBXII distribution with the pdf given in equation (1.5). The Lq-likelihood equations can be defined as

$$\sum_{i=1}^{n} [f(x_i;\alpha,c,k)]^{(1-q)} \bigtriangledown_{\alpha,c,k} \log f(x_i;\alpha,c,k) = 0.$$

Namely,

$$\sum_{i=1}^{n} \left[\frac{(\alpha ck)\, x_i^{c-1}(1+x_i^c)^{-(k+1)}}{[1-(1-\alpha)(1+x^c)^{-k}]^2}\right]^{(1-q)} \bigtriangledown_{\alpha,c,k} \log\left[\frac{(\alpha ck)\, x_i^{c-1}(1+x_i^c)^{-(k+1)}}{[1-(1-\alpha)(1+x^c)^{-k}]^2}\right] = 0 \tag{2.11}$$

where $\bigtriangledown_{\alpha,c,k} \log f(x_i;\alpha,c,k)$ is indicated the partial derivations of $\log f(x_i;\alpha,c,k)$ with respect to α, c and k. The $\log f(x_i;\alpha,c,k)$ of the MOEBXII distribution is

$$\log f(x_i;\alpha,c,k) = \log(\alpha ck)+(c-1)\log(x_i)-(k+1)\log(1+x_i^c)-2\log[1-(1-\alpha)(1+x^c)^{-k}] \tag{2.12}$$

and its derivations with respect to α, c and k are

$$\frac{\partial \log f(x_i;\alpha,c,k)}{\partial \alpha} = \frac{1}{\alpha} - 2\frac{(1+x_i^c)^{-k}}{1-(1-\alpha)\,(1+x_i^c)^{-k}}, \tag{2.13}$$

$$\frac{\partial \log f(x_i;\alpha,c,k)}{\partial c} = \frac{1}{c}+\log x_i-(k+1)\frac{x_i^c \log(x_i)}{(1+x_i^c)}-2k(1-\alpha)\frac{x_i^c(1+x_i^c)^{-(k+1)}\log(x_i)}{1-(1-\alpha)\,(1+x_i^c)^{-k}}, \tag{2.14}$$

$$\frac{\partial \log f(x_i;\alpha,c,k)}{\partial k} = \frac{1}{k} - \log(1+x_i^c) - 2(1-\alpha)\frac{(1+x_i^c)^{-k}\log(1+x_i^c)}{1-(1-\alpha)\,(1+x_i^c)^{-k}}. \tag{2.15}$$

When these equations are written in place of equation (2.11) it can be seen that there are no explicit forms for the MLq estimators of parameters α, c and k. However, these equations are rearranged as follows:

$$\widehat{\alpha} = \left(\frac{\sum_{i=1}^{n} w_i u_{1i}}{\sum_{i=1}^{n} w_i}\right)^{-1}, \quad \widehat{c} = \left(\frac{\sum_{i=1}^{n} w_i u_{2i}}{\sum_{i=1}^{n} w_i}\right)^{-1}, \quad \widehat{k} = \left(\frac{\sum_{i=1}^{n} w_i u_{3i}}{\sum_{i=1}^{n} w_i}\right)^{-1}$$

where

$$w_i = \left[\frac{(\alpha ck)\, x_i^{c-1}(1+x_i^c)^{-(k+1)}}{[1-(1-\alpha)(1+x^c)^{-k}]^2}\right]^{(1-q)},$$

$$u_{1i} = 2\frac{(1+x_i^c)^{-k}}{1-(1-\alpha)\,(1+x_i^c)^{-k}},$$

$$u_{2i}=-\log x_i+(k+1)\frac{x_i^c\log(x_i)}{(1+x_i^c)}+2k(1-\alpha)\frac{x_i^c(1+x_i^c)^{-(k+1)}\log(x_i)}{1-(1-\alpha)(1+x_i^c)^{-k}},$$

and

$$u_{3i}=\log(1+x_i^c)+2(1-\alpha)\frac{(1+x_i^c)^{-k}\log(1+x_i^c)}{1-(1-\alpha)(1+x_i^c)^{-k}}.$$

As it is seen from the above equations the MLq estimates can only be obtained by using numerical methods.

3. Simulation Study

In this simulation study, we consider the MOEBXII distribution when the sample size $n=25,50,100$. First, we generate data sets from the MOEBXII distribution using the following parameter values (α,c,k)=(3, 1, 1), (3, 1, 2), (3, 2, 1), (3, 2, 2), (3, 3, 3), (5, 1, 1), (5, 1, 2), (5, 2, 1), and (5, 2, 2). We use inverse of cumulative distribution function method to generate the data. We carry out two different simulation scenarios with and without outliers. We add 4% outliers, to show the performances of the MLq estimators under the presence of outliers. We compare the estimators using MSE values calculated as

$$MSE\left(\widehat{\theta}\right)=\frac{1}{1000}\sum_{i=1}^{1000}\left(\widehat{\theta}_i-\theta\right)^2$$

equation where θ=α,c and k. We repeate each simulation scenario 1000 times. For the robust estimators the tuning constants are taken as b_1=1,345 and b_2=3,5 for Huber's ρ and Tukey's ρ functions, respectivelly.

The calculated MSE values for each case and each method are summarized in Table 1-7.

Table 1 The MSE values of estimations for $n = 25$ without outliers

parameter	ML	Huber	Tukey	MLq
$\alpha = 3$	0.3452	0.0669	0.1382	0.0772
$c = 1$	0.1305	0.1999	0.2316	0.1485
$k = 1$	0.0755	0.0958	0.1006	0.1170
$\alpha = 3$	0.1694	0.0729	0.1029	0.2320
$c = 1$	0.0489	0.1186	0.1549	0.0656
$k = 2$	0.1154	0.1244	0.0914	0.2099
$\alpha = 3$	0.4775	0.1341	0.1374	0.0856
$c = 2$	0.2889	0.3690	0.4130	0.3386
$k = 1$	0.0682	0.1047	0.0920	0.1054
$\alpha = 3$	0.2483	0.095	0.1537	0.0854
$c = 2$	0.1861	0.3447	0.3839	0.2252
$k = 2$	0.1153	0.2034	0.1910	0.1789
$\alpha = 3$	0.3292	0.1869	0.2183	0.0832
$c = 3$	0.2523	0.4536	0.5261	0.3603
$k = 3$	0.1585	0.1802	0.3081	0.2875
$\alpha = 5$	0.1806	0.1812	0.1934	0.1969
$c = 1$	0.1531	0.3500	0.3286	0.1567
$k = 1$	0.0777	0.3641	0.4020	0.0905
$\alpha = 5$	0.0782	0.6678	0.7360	0.0868
$c = 1$	0.0671	0.1546	0.3173	0.0799
$k = 2$	0.1034	0.9243	0.9115	0.1293
$\alpha = 5$	0.3200	0.2032	0.1769	0.3905
$c = 2$	0.2837	0.3826	0.4714	0.3971
$k = 1$	0.0735	0.2884	0.3603	0.0969
$\alpha = 5$	0.2006	0.4720	0.6009	0.3891
$c = 2$	0.1983	0.3509	0.5652	0.2268
$k = 2$	0.0930	0.9028	0.9074	0.1193

In Table 1 we observe that, as expected the ML estimation generally gives smaller MSE values than the other estimation methods when there is no outlier in data set. Similar results can be observed from Table 2 and Table 3 for the larger sample sizes. Considering the MLq estimators, it is seen that they have smaller MSE values than the MSE values obtained from the M estimators for several simulation scenarios.

Table 2 The MSE values of Estimations for $n = 50$ without outliers

parameter	ML	Huber	Tukey	MLq
$\alpha = 3$	0.3137	0.0381	0.0579	0.1660
$c = 1$	0.0487	0.0975	0.1225	0.0559
$k = 1$	0.0464	0.0647	0.0714	0.0756
$\alpha = 3$	0.1627	0.0624	0.0968	0.8071
$c = 1$	0.0267	0.0729	0.0760	0.0343
$k = 2$	0.0513	0.0953	0.0785	0.0369
$\alpha = 3$	0.4070	0.0776	0.2079	0.7245
$c = 2$	0.2005	0.3308	0.3785	0.2111
$k = 1$	0.0459	0.0680	0.1163	0.0524
$\alpha = 3$	0.2398	0.1054	0.1632	0.1868
$c = 2$	0.1024	0.3106	0.3436	0.1489
$k = 2$	0.0873	0.1727	0.1546	0.1564
$\alpha = 3$	0.2032	0.1259	0.1765	0.2295
$c = 3$	0.1459	0.3531	0.3969	0.1811
$k = 3$	0.0942	0.2103	0.2564	0.2137
$\alpha = 5$	0.1720	0.1228	0.0747	0.1303
$c = 1$	0.0793	0.2887	0.2643	0.0818
$k = 1$	0.0543	0.3529	0.3687	0.0744
$\alpha = 5$	0.0828	0.6411	0.7622	0.1706
$c = 1$	0.0374	0.1250	0.2253	0.0478
$k = 2$	0.0680	0.9583	0.9501	0.1126
$\alpha = 5$	0.2503	0.2275	0.1358	0.3112
$c = 2$	0.2254	0.4446	0.5319	0.2481
$k = 1$	0.0435	0.3351	0.3745	0.0512
$\alpha = 5$	0.1992	0.6504	0.7152	0.3775
$c = 2$	0.1017	0.3216	0.5322	0.1144
$k = 2$	0.0694	0.9611	0.9458	0.1060

Table 3 The MSE values of estimations for $n = 100$ without outliers

parameter	ML	Huber	Tukey	MLq
$\alpha = 3$	0.4153	0.3535	0.3476	0.6458
$c = 1$	0.0276	0.0824	0.0892	0.0317
$k = 1$	0.0389	0.0582	0.0487	0.0516
$\alpha = 3$	0.2275	0.0457	0.0825	0.1889
$c = 1$	0.0138	0.0602	0.0771	0.0216
$k = 2$	0.0462	0.0551	0.0592	0.1194
$\alpha = 3$	0.4524	0.5477	0.1681	0.1372
$c = 2$	0.1018	0.2858	0.2966	0.1088
$k = 1$	0.0336	0.0574	0.0869	0.0461
$\alpha = 3$	0.3040	0.1612	0.1095	0.1420
$c = 2$	0.0601	0.1919	0.1927	0.0735
$k = 2$	0.0519	0.1422	0.1255	0.0995
$\alpha = 3$	0.1672	0.1656	0.2324	0.2268
$c = 3$	0.1045	0.4758	0.4858	0.1615
$k = 3$	0.0584	0.2141	0.3340	0.2093
$\alpha = 5$	0.3012	0.1272	0.0977	0.3613
$c = 1$	0.0342	0.2601	0.1918	0.0371
$k = 1$	0.0283	0.3552	0.3571	0.0402
$\alpha = 5$	0.1270	0.7229	0.7866	0.1623
$c = 1$	0.0133	0.1080	0.1235	0.0162
$k = 2$	0.0330	0.9778	0.9654	0.0634
$\alpha = 5$	0.3221	0.2279	0.1512	0.3692
$c = 2$	0.1428	0.4749	0.5141	0.1510
$k = 1$	0.0327	0.3462	0.3651	0.0380
$\alpha = 5$	0.1495	0.6411	0.7286	0.1660
$c = 2$	0.0390	0.3339	0.3970	0.0492
$k = 2$	0.0385	0.9640	0.9545	0.0662

It can be clearly seen that when n gets larger the MSE values for the MLq estimators become very similar to the MSE values of the ML estimators.

In Tables 4-6 we display the MSE values for the outlier case. We can clearly see that the performance of the ML estimators getting worse when there are some outliers in data. On the other hand, M and the MLq estimators are not affected from the outliers. It is observed that for most of the cases the MLq performs better than M estimation method. However for the parameter α the best results are obtained from M estimators, the second one is the MLq estimators and the worst one is the ML. For the other two parameters the MLq is generally the best.

Table 4 The MSE values of estimations for $n = 25$ with 4%outlier

parameter	ML	Huber	Tukey	MLq
$\alpha = 3$	0.7489	0.0593	0.0622	0.0874
$c = 1$	0.1945	0.1409	0.1722	0.1360
$k = 1$	0.1107	0.1237	0.1202	0.0942
$\alpha = 3$	0.7059	0.0830	0.1681	0.9232
$c = 1$	0.0826	0.0965	0.1287	0.0691
$k = 2$	0.3254	0.2627	0.1768	0.3216
$\alpha = 3$	1.4529	0.0867	0.1221	0.9047
$c = 2$	0.4233	0.3223	0.3467	0.2813
$k = 1$	0.1485	0.0990	0.1276	0.0876
$\alpha = 3$	1.1111	0.1009	0.2130	0.9902
$c = 2$	0.2293	0.1904	0.2919	0.1487
$k = 2$	0.4277	0.2306	0.2745	0.3320
$\alpha = 3$	1.5382	0.1876	0.2885	0.9202
$c = 3$	1.0872	0.3918	0.2272	0.3201
$k = 3$	1.0919	0.2282	0.2100	0.7891
$\alpha = 5$	0.6935	0.1157	0.1305	0.1924
$c = 1$	0.3365	0.3434	0.3270	0.2184
$k = 1$	0.1295	0.3749	0.4215	0.1092
$\alpha = 5$	0.6185	0.7697	0.8332	0.7949
$c = 1$	0.0565	0.0645	0.2501	0.0573
$k = 2$	0.1486	0.9690	0.9780	0.1611
$\alpha = 5$	0.9610	0.2171	0.1494	0.9461
$c = 2$	0.4329	0.3401	0.4641	0.3565
$k = 1$	0.0920	0.3302	0.4140	0.0724
$\alpha = 5$	0.8352	0.7546	0.8592	0.9907
$c = 2$	0.2941	0.1993	0.4597	0.2185
$k = 2$	0.1745	0.9627	0.9788	0.1768

Table 5 The MSE values of estimations for $n = 50$ with 4%outlier

parameter	ML	Huber	Tukey	MLq
$\alpha = 3$	0.9303	0.0171	0.0535	0.8253
$c = 1$	0.0583	0.0634	0.0925	0.0471
$k = 1$	0.0918	0.0985	0.1040	0.0724
$\alpha = 3$	0.8486	0.0565	0.1262	0.6551
$c = 1$	0.0417	0.0425	0.0652	0.0203
$k = 2$	0.2677	0.1759	0.1431	0.2709
$\alpha = 3$	1.5055	0.0596	0.1352	0.9860
$c = 2$	0.2480	0.2026	0.2306	0.1723
$k = 1$	0.1516	0.1026	0.1117	0.1082
$\alpha = 3$	1.0910	0.0810	0.1648	0.9737
$c = 2$	0.1585	0.1140	0.1685	0.0671
$k = 2$	0.4210	0.2257	0.2414	0.4241
$\alpha = 3$	1.6377	0.1504	0.3418	0.8956
$c = 3$	1.1001	0.4012	0.2323	0.5030
$k = 3$	1.1747	0.2207	0.2436	0.9830
$\alpha = 5$	0.8045	0.0985	0.1079	0.6132
$c = 1$	0.0869	0.2971	0.2235	0.0840
$k = 1$	0.0732	0.4027	0.4287	0.0665
$\alpha = 5$	0.9942	0.8671	0.9251	0.9813
$c = 1$	0.0354	0.0442	0.1458	0.0240
$k = 2$	0.1528	0.9921	0.9921	0.1070
$\alpha = 5$	1.2138	0.2063	0.1672	0.9699
$c = 2$	0.3928	0.3536	0.4320	0.2944
$k = 1$	0.0959	0.3580	0.4193	0.0809
$\alpha = 5$	1.1813	0.8155	0.9377	0.9334
$c = 2$	0.2029	0.1543	0.4593	0.1332
$k = 2$	0.2210	0.9854	0.9968	0.2159

Table 6 The MSE values of estimations for $n = 100$ with 4%outlier

parameter	ML	Huber	Tukey	MLq
$\alpha = 3$	1.4514	0.0148	0.0407	0.9564
$c = 1$	0.0473	0.0601	0.0737	0.0357
$k = 1$	0.1355	0.1087	0.0914	0.0977
$\alpha = 3$	1.5395	0.0584	0.1772	0.9886
$c = 1$	0.0236	0.0233	0.0301	0.0085
$k = 2$	0.5007	0.1676	0.1438	0.3297
$\alpha = 3$	1.7153	0.0801	0.1164	0.9999
$c = 2$	0.2369	0.2297	0.2152	0.1547
$k = 1$	0.1796	0.1050	0.1111	0.1265
$\alpha = 3$	1.7178	0.0486	0.1949	0.9719
$c = 2$	0.1165	0.1184	0.1338	0.0601
$k = 2$	0.6832	0.2235	0.2048	0.4849
$\alpha = 3$	1.9561	0.1498	0.3502	0.8962
$c = 3$	0.8292	0.2809	0.1503	0.7617
$k = 3$	1.6878	0.1453	0.1749	0.9762
$\alpha = 5$	1.1048	0.1095	0.0751	0.9545
$c = 1$	0.0508	0.2425	0.1662	0.0438
$k = 1$	0.0609	0.3916	0.4028	0.0541
$\alpha = 5$	1.2514	0.9246	0.9546	0.9902
$c = 1$	0.0280	0.0495	0.1389	0.0175
$k = 2$	0.1657	0.9937	0.9974	0.1436
$\alpha = 5$	1.5288	0.1684	0.1349	0.9978
$c = 2$	0.1333	0.3620	0.4251	0.1136
$k = 1$	0.0705	0.3712	0.4043	0.0589
$\alpha = 5$	1.5110	0.9092	0.9868	0.9985
$c = 2$	0.1463	0.1961	0.3593	0.1006
$k = 2$	0.2376	0.9976	0.9997	0.1925

To sum up the ML estimation method have better performance among the other methods when there are no outliers in the data according to the MSE values. However, the robust methods should be prefered if there are some outliers. Between the robust methods the MLq has better performance and since it is very similar to the ML, the MLq should be used instead of other robust methods.

Table 1-6 show the MSE values for the case $\alpha > 1$ to compare the MLq estimation with M estimation. Note that M estimation for the MOEBXII distribution is not defined for $\alpha < 1$. This is because for MOEBXII distribution, we have $\log(\alpha - 1)$ in Q function given in equation (2.5). On the other hand, we do not have this problem in MLq estimation. To compare the performance of the ML and the MLq estimators for the case $\alpha < 1$ we run a small simulation study with and without outlier. Table 7 gives the results for this simulation study.

Table 7 The MSE values of Estimations for $\alpha < 1$

		without outlier		4% outlier	
n	parameter	ML	MLq	ML	MLq
	$\alpha = 0.8$	0.1810	0.3929	0.4532	0.3261
25	$c = 0.5$	0.0174	0.0229	0.1976	0.0270
	$k = 0.8$	0.0970	0.2574	0.0953	0.0178
	$\alpha = 0.8$	0.1680	0.3007	0.1730	0.0246
50	$c = 0.5$	0.0075	0.0092	0.1162	0.0132
	$k = 0.8$	0.0792	0.1452	0.1299	0.0143
	$\alpha = 0.8$	0.1390	0.2128	0.2040	0.1763
100	$c = 0.5$	0.0045	0.0051	0.0065	0.0064
	$k = 0.8$	0.0567	0.0870	0.1530	0.1485
	$\alpha = 0.8$	0.0849	0.2813	0.1217	0.0263
25	$c = 0.5$	0.0116	0.0206	0.0137	0.0086
	$k = 3$	0.1080	0.6843	0.8337	0.2187
	$\alpha = 0.8$	0.0706	0.2381	0.2143	0.1625
50	$c = 0.5$	0.0046	0.0071	0.0078	0.0045
	$k = 3$	0.1260	0.6416	0.2622	0.1416
	$\alpha = 0.8$	0.0729	0.2220	0.1038	0.0744
100	$c = 0.5$	0.0030	0.0048	0.0084	0.0024
	$k = 3$	0.1475	0.5253	0.8342	0.5804
	$\alpha = 0.8$	0.1788	0.3274	0.2133	0.2091
25	$c = 3$	0.1864	0.3923	0.5838	0.5189
	$k = 0.5$	0.0374	0.0640	0.0767	0.0758
	$\alpha = 0.8$	0.1616	0.2645	0.2871	0.2713
50	$c = 3$	0.1823	0.2855	0.4097	0.2978
	$k = 0.5$	0.0340	0.0542	0.0996	0.0696
	$\alpha = 0.8$	0.1254	0.1658	0.3407	0.3260
100	$c = 3$	0.1310	0.1653	0.3375	0.2027
	$k = 0.5$	0.0242	0.0302	0.1194	0.1189

According to Table 7, MSE values of ML estimation for different values of parameters smaller than those of MLq when the data set has no outlier. However, if there are some outliers in the data the MLq estimation gives smaller MSE values than the ML for almost all the cases. These results show that the MLq estimation method produces better estimators than the ML estimation method in the case of outliers.

4. Real Data Example

Global warming is a vital issue in recent years and the major reason for it is the man-made greenhouse gas emissions which mainly include carbon dioxide (CO_2), methane (CH_4), and nitrous oxide (N_20). In this example we will use the

CO_2 emission data for 34 countries [5]. This data set is obtained from Eurostats database. The data were recorded for the year 2015. The histogram of the data set, which is given in Figure 1, shows moderated skewness to the right and includes one outlier. Therefore, skew distributions would be plausible to model this data set. In this paper, we will use the MOEBXII distribution to model this data set. There are two reasons to use the MOEBXII distribution. One is the data has positive support which is convenient to use the MOEBXII distribution and the other is it is very flexible to catch the shape of the data.

We estimate the unknown parameters of the distribution using the ML,the MLq and robust M estimation methods proposed by [15]. The results are given in Table 8. The fitted densities obtained from the ML, the MLq and robust estimates are shown in Figure 1.

Table 8 Parameter Estimations for the CO_2emissions

	ML	MLq	Huber	Tukey
$\widehat{\alpha}$	18.5395	20.8525	20.8976	20.8975
$\widehat{c}$	7.0058	4.3334	4.2784	4.2784
$\widehat{k}$	0.5310	0.7497	0.6948	0.6997

From Figure 1, we observed that the best fit is obtained from the MLq method. It is followed by the fitted density obtained from Huber M estimates. While the ML method gives overfitted density, the robust estimates obtained from Tukey ρ function present underestimated fit.

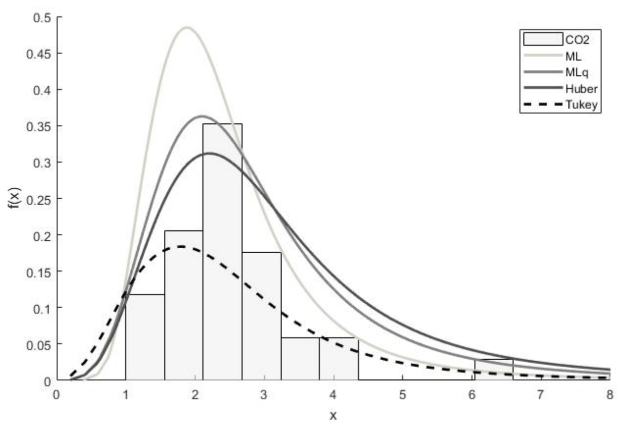

FIGURE 1. Histogram of the CO_2 emmisions and pdf fits with estimated parameters

5. Conclusion

MOEBXII distribution would be a good alternative to the distributions that are used in economics, reliability, survival analysis and so on. The parameters of this distribution have been estimated using ML and Bayesian methods [2]. Recently, robust methods have also been used to estimate the parameters [15]. ML methods are very sensitive to the outliers. Therefore robust methods should be used to estimate the parameters. However, since robust methods used in [15] cannot be compitable for the case $\alpha < 1$, alternative robust methods are needed to estimate the parameters. In this paper, we have used the MLq estimation method to estimate the parameters of the MOEBXII distribution. We have carried out a small simulation study and a real data example to show that the MLq method could be a good alternative estimation method to the existing ones. Our simulation results also reveal that the MLq estimates are better than ML in most simulation scenarios and it is compatable with the robust methods. Further, unlike the other robust methods the MLq could also be used for the case $\alpha < 1$. Concerning the real data example, we observe that the best fit is obtained from the MLq method. To sum up, the MLq estimation method could be used to estimate the parameters of MOEBXII distribution and the estimators would be robust to the outliers in the data.

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Current address: Şenay Özdemir: Afyon Kocatepe University, Faculty of Literature and Sciences, Department of Statistics 03200 Gazlıgol Yolu, AFYONKARAHİSAR TURKEY.

E-mail address: senayozdemir@aku.edu.tr

ORCID Address: http://orcid.org/0000-0003-2726-2169

Current address: Yeşim Güney: Ankara University, Faculty of Sciences, Department of Statistics 06100 Besevler ANKARA TURKEY.

E-mail address: ydone@ankara.edu.tr

ORCID Address: http://orcid.org/0000-0003-2316-8888

Current address: Yetkin Tuaç: Ankara University, Faculty of Sciences, Department of Statistics 06100 Besevler ANKARA TURKEY.

E-mail address: ytuac@ankara.edu.tr

ORCID Address: http://orcid.org/0000-0001-6590-3233

Current address: Olcay Arslan: Ankara University, Faculty of Sciences, Department of Statistics 06100 Besevler ANKARA TURKEY.

E-mail address: oarslan@ankara.edu.tr

ORCID Address: http://orcid.org/0000-0002-7067-4997