



## GLOBAL BEHAVIOR OF A THREE-DIMENSIONAL SYSTEM OF DIFFERENCE EQUATIONS OF ORDER THREE

DURHASAN TURGUT TOLLU AND İBRAHİM YALÇINKAYA

ABSTRACT. In this paper, we investigate the global behavior of the positive solutions of the system of difference equations

$$u_{n+1} = \frac{\alpha_1 u_{n-1}}{\beta_1 + \gamma_1 v_{n-2}^p}, \quad v_{n+1} = \frac{\alpha_2 v_{n-1}}{\beta_2 + \gamma_2 w_{n-2}^q}, \quad w_{n+1} = \frac{\alpha_3 w_{n-1}}{\beta_3 + \gamma_3 u_{n-2}^r}$$

for  $n \in \mathbb{N}_0$  where the initial conditions  $u_{-i}, v_{-i}, w_{-i}$  ( $i = 0, 1, 2$ ) are non-negative real numbers and the parameters  $\alpha_j, \beta_j, \gamma_j$  ( $j = 1, 2, 3$ ) and  $p, q, r$  are positive real numbers, by extending some results in the literature.

### 1. INTRODUCTION

Recently, difference equations have gained a great importance. Most of the recent applications of these equations have appeared in many scientific areas such as biology, physics, engineering, economics. Particularly, rational difference equations and their systems have great importance in applications. So, it is very worthy to examine the behavior of solutions of a system of rational difference equations and to discuss the stability character of their equilibrium points. In recent years, many researchers have investigated global behavior of solutions of difference equations or their two dimensional systems and have suggested some diverse methods for the qualitative behavior of the their solutions. But, studies on three dimensional systems of difference equations in the literature are quite limited. For example, Kulenović and Nurkanović [12] studied the global asymptotic behavior of solutions of the system of difference equations

$$x_{n+1} = \frac{a + x_n}{b + y_n}, \quad y_{n+1} = \frac{c + y_n}{d + z_n}, \quad z_{n+1} = \frac{e + z_n}{f + x_n}, \quad n \in \mathbb{N}_0,$$

where  $a, b, c, d, e, f \in (0, \infty)$  and the initial conditions  $x_0, y_0$  and  $z_0$  are arbitrary non-negative numbers. Kurbanli [15] studied the behavior of solutions of the system

---

Received by the editors: January 01, 2017, Accepted: November 27, 2017.

2010 *Mathematics Subject Classification.* 39A10.

*Key words and phrases.* System of difference equations; stability; global behavior; periodic solution.

©2018 Ankara University.  
 Communications Faculty of Sciences University of Ankara-Series A1 Mathematics and Statistics.  
 Communications de la Faculté des Sciences de l'Université d'Ankara-Séries A1 Mathématiques et Statistiques.

of rational difference equations

$$x_{n+1} = \frac{x_{n-1}}{y_n x_{n-1} - 1}, \quad y_{n+1} = \frac{y_{n-1}}{x_n y_{n-1} - 1}, \quad z_{n+1} = \frac{z_{n-1}}{y_n z_{n-1} - 1}, \quad n \in \mathbb{N}_0,$$

where the initial conditions  $x_{-1}, x_0, y_{-1}, y_0, z_{-1}, z_0$  are real numbers. See also [13, 14, 16]. Yazlik et al. [34] obtained the explicit solutions of a three-dimensional system of difference equations with multiplicative terms

$$x_{n+1} = \frac{x_n y_{n-1}}{a_0 x_n + b_0 y_{n-2}}, \quad y_{n+1} = \frac{y_n z_{n-1}}{a_1 y_n + b_1 z_{n-2}}, \quad z_{n+1} = \frac{z_n x_{n-1}}{a_2 z_n + b_2 x_{n-2}}, \quad n \in \mathbb{N}_0,$$

where the parameters  $a_i, b_i$ , and the initial conditions  $x_{-i}, y_{-i}$  ( $i = 0, 1, 2$ ) are real numbers. extending some results in literature. Also, by using explicit forms of the solutions, they studied the asymptotic behavior of well-defined solutions of the system. For more works related to difference equations and their two and three dimensional systems, see references [1, 2, 3, 5, 6, 4, 9, 17, 18, 19, 20, 21, 22, 23, 24, 25, 26, 27, 28, 30, 29, 31, 32, 33, 34].

In [7], El-Owaidy et al. investigated global behavior of the difference equation

$$x_{n+1} = \frac{\alpha x_{n-1}}{\beta + \gamma x_{n-2}^p}, \quad n \in \mathbb{N}_0, \quad (1)$$

with non-negative parameters and non-negative initial conditions. Gumus and Soykan [10] studied the dynamic behavior of the positive solutions for a system of rational difference equations of the following form

$$u_{n+1} = \frac{\alpha u_{n-1}}{\beta + \gamma v_{n-2}^p}, \quad v_{n+1} = \frac{\alpha_1 v_{n-1}}{\beta_1 + \gamma_1 u_{n-2}^p}, \quad n \in \mathbb{N}_0, \quad (2)$$

where the parameters and initial conditions are positive real numbers.

In the present paper, we investigate the global behavior of the positive solutions of the system of difference equations

$$u_{n+1} = \frac{\alpha_1 u_{n-1}}{\beta_1 + \gamma_1 v_{n-2}^p}, \quad v_{n+1} = \frac{\alpha_2 v_{n-1}}{\beta_2 + \gamma_2 w_{n-2}^q}, \quad w_{n+1} = \frac{\alpha_3 w_{n-1}}{\beta_3 + \gamma_3 u_{n-2}^r}, \quad n \in \mathbb{N}_0, \quad (3)$$

where the initial conditions  $u_{-i}, v_{-i}, w_{-i}$  ( $i = 0, 1, 2$ ) are non-negative real numbers and the parameters  $\alpha_j, \beta_j, \gamma_j$  ( $j = 1, 2, 3$ ) and  $p, q, r$  are positive real numbers, by extending some results in the literature. System (3) is a natural extension of Eq. (1) and system (2). Note that system (3) can be written as

$$x_{n+1} = \frac{ax_{n-1}}{1 + y_{n-2}^p}, \quad y_{n+1} = \frac{by_{n-1}}{1 + z_{n-2}^q}, \quad z_{n+1} = \frac{cz_{n-1}}{1 + x_{n-2}^r}, \quad n \in \mathbb{N}_0, \quad (4)$$

by the change of variables  $u_n = \left(\frac{\beta_3}{\gamma_3}\right)^{1/r} x_n$ ,  $v_n = \left(\frac{\beta_1}{\gamma_1}\right)^{1/p} y_n$ ,  $w_n = \left(\frac{\beta_2}{\gamma_2}\right)^{1/q} z_n$  with  $a = \frac{\alpha_1}{\beta_1}$ ,  $b = \frac{\alpha_2}{\beta_2}$  and  $c = \frac{\alpha_3}{\beta_3}$ . So, we will consider system (4) instead of system (3) from now.

## 2. PRELIMINARIES

Let  $I_1, I_2, I_3$  be some intervals of real numbers and  $f : I_1^{k+1} \times I_2^{k+1} \times I_3^{k+1} \rightarrow I_1$ ,  $g : I_1^{k+1} \times I_2^{k+1} \times I_3^{k+1} \rightarrow I_2$ ,  $h : I_1^{k+1} \times I_2^{k+1} \times I_3^{k+1} \rightarrow I_3$  be continuously differentiable functions. Then, for every initial conditions  $(x_{-i}, y_{-i}, z_{-i}) \in I_1 \times I_2 \times I_3$ ,  $i = 0, \dots, k$ , the system of difference equations

$$\begin{cases} x_{n+1} = f(x_n, \dots, x_{n-k}, y_n, \dots, y_{n-k}, z_n, \dots, z_{n-k}) \\ y_{n+1} = g(x_n, \dots, x_{n-k}, y_n, \dots, y_{n-k}, z_n, \dots, z_{n-k}) \\ z_{n+1} = h(x_n, \dots, x_{n-k}, y_n, \dots, y_{n-k}, z_n, \dots, z_{n-k}) \end{cases} \quad \text{for } n \in \mathbb{N}_0 \quad (5)$$

has the unique solution  $\{(x_n, y_n, z_n)\}_{n=-k}^{\infty}$ . Also, an equilibrium point of system (5) is a point  $(\bar{x}, \bar{y}, \bar{z})$  that satisfies

$$\begin{aligned} \bar{x} &= f(\bar{x}, \dots, \bar{x}, \bar{y}, \dots, \bar{y}, \bar{z}, \dots, \bar{z}), \\ \bar{y} &= g(\bar{x}, \dots, \bar{x}, \bar{y}, \dots, \bar{y}, \bar{z}, \dots, \bar{z}), \\ \bar{z} &= h(\bar{x}, \dots, \bar{x}, \bar{y}, \dots, \bar{y}, \bar{z}, \dots, \bar{z}), \end{aligned}$$

We rewrite system (5) in the vector form

$$X_{n+1} = F(X_n), \quad n \in \mathbb{N}_0, \quad (6)$$

where  $X_n = (x_n, \dots, x_{n-k}, y_n, \dots, y_{n-k}, z_n, \dots, z_{n-k})^T$ ,  $F$  is a vector map such that  $F : I_1^{k+1} \times I_2^{k+1} \times I_3^{k+1} \rightarrow I_1^{k+1} \times I_2^{k+1} \times I_3^{k+1}$  and

$$F \begin{pmatrix} x_0 \\ \vdots \\ x_k \\ y_0 \\ \vdots \\ y_k \\ z_0 \\ \vdots \\ z_k \end{pmatrix} = \begin{pmatrix} f(x_0, \dots, x_k, y_0, \dots, y_k, z_0, \dots, z_k) \\ \vdots \\ x_{k-1} \\ g(x_0, \dots, x_k, y_0, \dots, y_k, z_0, \dots, z_k) \\ \vdots \\ y_{k-1} \\ h(x_0, \dots, x_k, y_0, \dots, y_k, z_0, \dots, z_k) \\ \vdots \\ z_{k-1} \end{pmatrix}.$$

It is clear that if an equilibrium point of system (5) is  $(\bar{x}, \bar{y}, \bar{z})$ , then the corresponding equilibrium point of system (6) is the point  $\bar{X} = (\bar{x}, \dots, \bar{x}, \bar{y}, \dots, \bar{y}, \bar{z}, \dots, \bar{z})^T$ .

In this study, we denote by  $\|\cdot\|$  any convenient vector norm and the corresponding matrix norm. Also, we denote by  $X_0 \in I_1^{k+1} \times I_2^{k+1} \times I_3^{k+1}$  a initial condition of system (6).

**Definition 1.** Let  $\bar{X}$  be an equilibrium point of system (6). Then,

- i) The equilibrium point  $\bar{X}$  is called stable if for every  $\epsilon > 0$  there exists  $\delta > 0$  such that  $\|X_0 - \bar{X}\| < \delta$  implies  $\|X_n - \bar{X}\| < \epsilon$ , for all  $n \geq 0$ . Otherwise the equilibrium point  $\bar{X}$  is called unstable.

- ii) The equilibrium point  $\bar{X}$  is called *local asymptotically stable* if it is stable and there exists  $\gamma > 0$  such that  $\|X_0 - \bar{X}\| < \gamma$  and  $X_n \rightarrow \bar{X}$  as  $n \rightarrow \infty$ .
- iii) The equilibrium point  $\bar{X}$  is called a *global attractor* if  $X_n \rightarrow \bar{X}$  as  $n \rightarrow \infty$ .
- iv) The equilibrium point  $\bar{X}$  is called *globally asymptotically stable* if it is both local asymptotically stable and global attractor.

The linearized system of system (6) evaluated at the equilibrium point  $\bar{X}$  is

$$Z_{n+1} = J_F Z_n, \quad n \in \mathbb{N}_0, \quad (7)$$

where  $J_F$  is the Jacobian matrix of the map  $F$  at the equilibrium point  $\bar{X}$ . The characteristic polynomial of system (7) about the equilibrium point  $\bar{X}$  is

$$P(\lambda) = a_0 \lambda^{3(k+1)} + a_1 \lambda^{3k+2} + \dots + a_{3k+2} \lambda + a_{3(k+1)}, \quad (8)$$

with real coefficients and  $a_0 > 0$ .

**Theorem 2.** [11] *Assume that  $\bar{X}$  is a equilibrium point of system (6). If all eigenvalues of the Jacobian matrix  $J_F$  evaluated at  $\bar{X}$  lie in the open unit disk  $|\lambda| < 1$ , then  $\bar{X}$  is locally asymptotically stable. If one of them has a modulus greater than one, then  $\bar{X}$  is unstable.*

### 3. STABILITY OF THE SYSTEM

In this section, we investigate the stability of the equilibrium points of system (4). When  $a, b, c \in (0, 1)$ , the point  $(\bar{x}_1, \bar{y}_1, \bar{z}_1) = (0, 0, 0)$  is the unique nonnegative equilibrium point of system (4). When  $a, b, c \in (1, \infty)$ , the unique positive equilibrium point of system (4) is

$$(\bar{x}_2, \bar{y}_2, \bar{z}_2) = \left( (c-1)^{1/r}, (a-1)^{1/p}, (b-1)^{1/q} \right).$$

In addition,

- (i) if  $a = b = c = 1$ , then

$$(\bar{x}_3, \bar{y}_3, \bar{z}_3) = (c_1, 0, 0), \quad (\bar{x}_4, \bar{y}_4, \bar{z}_4) = (0, c_2, 0) \quad \text{and} \quad (\bar{x}_5, \bar{y}_5, \bar{z}_5) = (0, 0, c_3),$$

- (ii) if  $a = 1$  and  $b, c \in (1, \infty)$ , then

$$(\bar{x}_1, \bar{y}_1, \bar{z}_1) = (0, 0, 0) \quad \text{and} \quad (\bar{x}_6, \bar{y}_6, \bar{z}_6) = \left( (c-1)^{1/r}, 0, c_3 \right),$$

- (iii) if  $a \neq 1$ ,  $b = 1$  and  $c \in (1, \infty)$ , then

$$(\bar{x}_1, \bar{y}_1, \bar{z}_1) = (0, 0, 0) \quad \text{and} \quad (\bar{x}_7, \bar{y}_7, \bar{z}_7) = \left( c_1, (a-1)^{1/p}, 0 \right),$$

- (iv) if  $a, b \in (1, \infty)$  and  $c = 1$ , then

$$(\bar{x}_1, \bar{y}_1, \bar{z}_1) = (0, 0, 0) \quad \text{and} \quad (\bar{x}_8, \bar{y}_8, \bar{z}_8) = \left( 0, c_2, (b-1)^{1/q} \right),$$

- (v) if  $a = b = 1$  and  $c \in (1, \infty)$ , then

$$(\bar{x}_3, \bar{y}_3, \bar{z}_3) = (c_1, 0, 0), \quad (\bar{x}_4, \bar{y}_4, \bar{z}_4) = (0, c_2, 0) \quad \text{and} \quad (\bar{x}_6, \bar{y}_6, \bar{z}_6) = \left( (c-1)^{1/r}, 0, c_3 \right),$$

(vi) if  $a = c = 1$  and  $b \in (1, \infty)$ , then

$$(\bar{x}_3, \bar{y}_3, \bar{z}_3) = (c_1, 0, 0), \quad (\bar{x}_5, \bar{y}_5, \bar{z}_5) = (0, 0, c_3) \quad \text{and} \quad (\bar{x}_8, \bar{y}_8, \bar{z}_8) = (0, c_2, (b-1)^{1/q}),$$

(vii) if  $b = c = 1$  and  $a \in (1, \infty)$ , then

$$(\bar{x}_4, \bar{y}_4, \bar{z}_4) = (0, c_2, 0), \quad (\bar{x}_5, \bar{y}_5, \bar{z}_5) = (0, 0, c_3) \quad \text{and} \quad (\bar{x}_7, \bar{y}_7, \bar{z}_7) = (c_1, (a-1)^{1/p}, 0),$$

where  $c_1, c_2$  and  $c_3$  are real numbers.

**Theorem 3.** *The following statements hold:*

- i) *If  $a, b, c \in (0, 1)$ , then the equilibrium point  $(\bar{x}_1, \bar{y}_1, \bar{z}_1) = (0, 0, 0)$  of system (4) is locally asymptotically stable.*
- ii) *If  $a, b, c \in (1, \infty)$ , then the equilibrium point  $(\bar{x}_1, \bar{y}_1, \bar{z}_1) = (0, 0, 0)$  of system (4) is unstable.*
- iii) *If  $a, b, c \in (1, \infty)$ , then the positive equilibrium point  $(\bar{x}_2, \bar{y}_2, \bar{z}_2) = ((c-1)^{1/r}, (a-1)^{1/p}, (b-1)^{1/q})$  of system (4) is unstable.*

*Proof.* First, we can write system (4) in the form of system (6) such that

$$X_n = (x_n, x_{n-1}, x_{n-2}, y_n, y_{n-1}, y_{n-2}, z_n, z_{n-1}, z_{n-2})^T$$

the map  $F$  is

$$F \begin{pmatrix} \begin{pmatrix} x_0 \\ x_1 \\ x_2 \\ y_0 \\ y_1 \\ y_2 \\ z_0 \\ z_1 \\ z_2 \end{pmatrix} \end{pmatrix} = \begin{pmatrix} ax_1/(1+y_2^p) \\ x_0 \\ x_1 \\ by_1/(1+z_2^q) \\ y_0 \\ y_1 \\ cz_1/(1+x_2^r) \\ z_0 \\ z_1 \end{pmatrix}.$$

- i) The linearized system of (4) about the equilibrium point  $\bar{X}_0 = (0, 0, 0, 0, 0, 0, 0, 0, 0)^T$  is given

$$X_{n+1} = J_F(\bar{X}_0)X_n,$$

where

$$X_n = \begin{pmatrix} x_n \\ x_{n-1} \\ x_{n-2} \\ y_n \\ y_{n-1} \\ y_{n-2} \\ z_n \\ z_{n-1} \\ z_{n-2} \end{pmatrix} \quad \text{and} \quad J_F(\bar{X}_0) = \begin{pmatrix} 0 & a & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & b & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & c & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}.$$

The characteristic equation of  $J_F(\bar{X}_0)$  is given by

$$P(\lambda) = \lambda^9 - (a + b + c)\lambda^7 + (ab + ac + bc)\lambda^5 - abc\lambda^3 = 0 \quad (9)$$

or

$$P(\lambda) = \lambda^3 (\lambda^2 - a) (\lambda^2 - b) (\lambda^2 - c) = 0.$$

It is easy to see that if  $a, b, c \in (0, 1)$ , then all the roots of the characteristic equation (9) lie in the open unit disk  $|\lambda| < 1$ . So, the equilibrium point  $(\bar{x}_1, \bar{y}_1, \bar{z}_1) = (0, 0, 0)$  of (4) is locally asymptotically stable.

- ii) It is clearly seen that if  $a, b, c \in (1, \infty)$ , then some roots of characteristic equation (9) have absolute value greater than one. In this case, the equilibrium point  $(\bar{x}_1, \bar{y}_1, \bar{z}_1) = (0, 0, 0)$  of (4) is unstable.
- iii) The linearized system of (4) about the positive equilibrium point

$$\bar{X}_{a,b,c} = \begin{pmatrix} (c-1)^{1/r} \\ (c-1)^{1/r} \\ (c-1)^{1/r} \\ (a-1)^{1/p} \\ (a-1)^{1/p} \\ (a-1)^{1/p} \\ (b-1)^{1/q} \\ (b-1)^{1/q} \\ (b-1)^{1/q} \end{pmatrix}$$

is given by

$$X_{n+1} = J_F(\bar{X}_{a,b,c})X_n,$$

where

$$X_n = \begin{pmatrix} x_n \\ x_{n-1} \\ x_{n-2} \\ y_n \\ y_{n-1} \\ y_{n-2} \\ z_n \\ z_{n-1} \\ z_{n-2} \end{pmatrix}, J_F(\bar{X}_{a,b,c}) = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & A & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & B \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & C & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix},$$

with

$$A = -\frac{p(a-1)^{(p-1)/p}(c-1)^{1/r}}{a}, B = -\frac{q(b-1)^{(q-1)/q}(a-1)^{1/p}}{b}$$

and

$$C = -\frac{r(c-1)^{(r-1)/r}(b-1)^{1/q}}{c}.$$

The characteristic polynomial of  $J_F(\bar{X}_{a,b,c})$  is given by

$$P(\lambda) = \lambda^9 - 3\lambda^7 + 3\lambda^5 - \lambda^3 + \frac{pqr(a-1)(b-1)(c-1)}{abc}. \quad (10)$$

It is clear that  $P(\lambda)$  has a root in interval  $(-\infty, -1)$ , since

$$P(-1) = \frac{pqr(a-1)(b-1)(c-1)}{abc} > 0 \text{ and } \lim_{\lambda \rightarrow -\infty} P(\lambda) = -\infty.$$

So, from Theorem 2, we can say that if  $a, b, c \in (1, \infty)$ , then the positive equilibrium point  $(\bar{x}_2, \bar{y}_2, \bar{z}_2) = \left( (c-1)^{1/r}, (a-1)^{1/p}, (b-1)^{1/q} \right)$  of system (4) is unstable.  $\square$

**Theorem 4.** *If  $a, b, c \in (0, 1)$ , then the equilibrium point  $(\bar{x}_1, \bar{y}_1, \bar{z}_1) = (0, 0, 0)$  of system (4) is globally asymptotically stable.*

*Proof.* From Theorem 3, we know that if  $a, b, c \in (0, 1)$ , then the equilibrium point  $(\bar{x}_1, \bar{y}_1, \bar{z}_1) = (0, 0, 0)$  of system (4) is locally asymptotically stable. Hence, it suffices to show that

$$\lim_{n \rightarrow \infty} (x_n, y_n, z_n) = (0, 0, 0). \quad (11)$$

From system (4), we have that

$$0 \leq x_{n+1} = \frac{ax_{n-1}}{1+y_{n-2}^p} \leq ax_{n-1}, \quad 0 \leq y_{n+1} = \frac{by_{n-1}}{1+z_{n-2}^q} \leq by_{n-1}, \quad 0 \leq z_{n+1} = \frac{cz_{n-1}}{1+x_{n-2}^r} \leq cz_{n-1} \quad (12)$$

for  $n \in \mathbb{N}_0$ . From (12), we have by induction

$$0 \leq x_{2n-i} \leq a^n x_{-i}, \quad 0 \leq y_{2n-i} \leq b^n y_{-i}, \quad 0 \leq z_{2n-i} \leq c^n z_{-i} \quad (13)$$

where  $x_{-i}, y_{-i}, z_{-i}$  ( $i = 0, 1$ ) are the initial conditions. Consequently, by taking limits of inequalities in (13) when  $a, b, c \in (0, 1)$ , then we have the limit in (11) which completes the proof.  $\square$

#### 4. OSCILLATION BEHAVIOR AND EXISTENCE OF UNBOUNDED SOLUTIONS

In the following result, we are concerned with the oscillation of positive solutions of system (4) about the equilibrium point  $(\bar{x}_2, \bar{y}_2, \bar{z}_2) = \left( (c-1)^{1/r}, (a-1)^{1/p}, (b-1)^{1/q} \right)$ .

**Theorem 5.** *Assume that  $a, b, c \in (1, \infty)$  and let  $\{(x_n, y_n, z_n)\}_{n=-2}^{\infty}$  be a positive solution of system (4) such that*

$$x_{-2}, x_0 \geq \bar{x}_2, \quad x_{-1} < \bar{x}_2, \quad y_{-2}, y_0 \geq \bar{y}_2, \quad y_{-1} < \bar{y}_2, \quad z_{-2}, z_0 \geq \bar{z}_2, \quad z_{-1} < \bar{z}_2 \quad (14)$$

or

$$x_{-2}, x_0 < \bar{x}_2, \quad x_{-1} \geq \bar{x}_2, \quad y_{-2}, y_0 < \bar{y}_2, \quad y_{-1} \geq \bar{y}_2, \quad z_{-2}, z_0 < \bar{z}_2, \quad z_{-1} \geq \bar{z}_2. \quad (15)$$

Then,  $\{(x_n, y_n, z_n)\}_{n=-2}^{\infty}$  oscillates about the equilibrium point  $(\bar{x}_2, \bar{y}_2, \bar{z}_2)$  with semi-cycles of length one.

*Proof.* Assume that (14) holds. (The case where (15) holds is similar and will be omitted.) From (4), we have

$$\begin{aligned} x_1 &= \frac{ax_{-1}}{1+y_{-2}^p} < \frac{a\bar{x}_2}{1+\bar{y}_2^p} = \bar{x}_2, \\ y_1 &= \frac{by_{-1}}{1+z_{-2}^q} < \frac{b\bar{y}_2}{1+\bar{z}_2^q} = \bar{y}_2, \\ z_1 &= \frac{cz_{-1}}{1+x_{-2}^r} < \frac{c\bar{z}_2}{1+\bar{x}_2^r} = \bar{z}_2 \end{aligned}$$

and

$$\begin{aligned} x_2 &= \frac{ax_0}{1+y_{-1}^p} \geq \frac{a\bar{x}_2}{1+\bar{y}_2^p} = \bar{x}_2, \\ y_2 &= \frac{by_0}{1+z_{-1}^q} \geq \frac{b\bar{y}_2}{1+\bar{z}_2^q} = \bar{y}_2, \\ z_2 &= \frac{cz_0}{1+x_{-1}^r} \geq \frac{c\bar{z}_2}{1+\bar{x}_2^r} = \bar{z}_2 \end{aligned}$$

then, the proof follows by induction.  $\square$

In the following theorem, we show the existence of unbounded solutions for system (4).

**Theorem 6.** *Assume that  $a, b, c \in (1, \infty)$ , then system (4) possesses an unbounded solution.*

*Proof.* From Theorem 5, we can assume that without loss of generality that the solution  $\{(x_n, y_n, z_n)\}_{n=-2}^{\infty}$  of system (4) is such that  $x_{2n-1} < \bar{x}_2$ ,  $y_{2n-1} < \bar{y}_2$ ,  $z_{2n-1} < \bar{z}_2$ ,  $x_{2n} > \bar{x}_2$ ,  $y_{2n} > \bar{y}_2$  and  $z_{2n} > \bar{z}_2$  for  $n \in \mathbb{N}_0$ . Then, we have

$$\begin{aligned} x_{2n+2} &= \frac{ax_{2n}}{1+y_{2n-1}^p} > \frac{ax_{2n}}{1+\bar{y}_2^p} = \frac{ax_{2n}}{1+(a-1)} = x_{2n}, \\ y_{2n+2} &= \frac{by_{2n}}{1+z_{2n-1}^q} > \frac{by_{2n}}{1+\bar{z}_2^q} = \frac{by_{2n}}{1+(b-1)} = y_{2n}, \\ z_{2n+2} &= \frac{cz_{2n}}{1+x_{2n-1}^r} > \frac{cz_{2n}}{1+\bar{x}_2^r} = \frac{cz_{2n}}{1+(c-1)} = z_{2n} \end{aligned}$$

and

$$\begin{aligned} x_{2n+3} &= \frac{ax_{2n+1}}{1+y_{2n}^p} < \frac{ax_{2n+1}}{1+\bar{y}_2^p} = \frac{ax_{2n+1}}{1+(a-1)} = x_{2n+1}, \\ y_{2n+3} &= \frac{by_{2n+1}}{1+z_{2n}^q} < \frac{by_{2n+1}}{1+\bar{z}_2^q} = \frac{by_{2n+1}}{1+(b-1)} = y_{2n+1}, \\ z_{2n+3} &= \frac{cz_{2n+1}}{1+x_{2n}^r} < \frac{cz_{2n+1}}{1+\bar{x}_2^r} = \frac{cz_{2n+1}}{1+(c-1)} = z_{2n+1} \end{aligned}$$



from which it follows that

$$\lim_{n \rightarrow \infty} (x_{2n}, y_{2n}, z_{2n}) = (\infty, \infty, \infty) \quad \text{and} \quad \lim_{n \rightarrow \infty} (x_{2n+1}, y_{2n+1}, z_{2n+1}) = (0, 0, 0)$$

which completes the proof.  $\square$

## 5. PERIODICITY

In this section, we investigate the existence of period two solution of system (4).

**Theorem 7.** *If  $a = b = c = 1$ , then, system (4) possesses the prime period two solution*

$$\dots, (0, 0, \varphi), (0, 0, \psi), (0, 0, \varphi), (0, 0, \psi), \dots$$

with  $\varphi, \psi > 0$ . Furthermore, every solution of system (4) converges to a period two solution.

*Proof.* Assume that  $a = b = c = 1$  and let  $\{(x_n, y_n, z_n)\}_{n=-2}^{\infty}$  be a solution of system (4). Then, from system (4), we have

$$\begin{aligned} x_{2n+1} &= \frac{x_{2n-1}}{1+y_{2n-2}^p} \quad \text{and} \quad x_{2n+2} = \frac{x_{2n}}{1+y_{2n-1}^p} \\ y_{2n+1} &= \frac{y_{2n-1}}{1+z_{2n-2}^q} \quad \text{and} \quad y_{2n+2} = \frac{y_{2n}}{1+z_{2n-1}^q} \\ z_{2n+1} &= \frac{z_{2n-1}}{1+x_{2n-2}^r} \quad \text{and} \quad z_{2n+2} = \frac{z_{2n}}{1+x_{2n-1}^r} \end{aligned} \quad (16)$$

for  $n \in \mathbb{N}_0$ . From (16), we get

$$\begin{aligned} x_{2n-1} &= x_{-1} \prod_{i=0}^{n-1} \left( \frac{1}{1+y_{2i-2}^p} \right) \quad \text{and} \quad x_{2n} = x_0 \prod_{i=0}^{n-1} \left( \frac{1}{1+y_{2i-1}^p} \right) \\ y_{2n-1} &= y_{-1} \prod_{i=0}^{n-1} \left( \frac{1}{1+z_{2i-2}^q} \right) \quad \text{and} \quad y_{2n} = y_0 \prod_{i=0}^{n-1} \left( \frac{1}{1+z_{2i-1}^q} \right) \\ z_{2n-1} &= z_{-1} \prod_{i=0}^{n-1} \left( \frac{1}{1+x_{2i-2}^r} \right) \quad \text{and} \quad z_{2n} = z_0 \prod_{i=0}^{n-1} \left( \frac{1}{1+x_{2i-1}^r} \right) \end{aligned} \quad (17)$$

for  $n \in \mathbb{N}_0$ . If  $(x_{-1}, x_0) = (0, 0)$  and  $(y_{-1}, y_0) = (0, 0)$ , then  $(x_n, y_n) = (0, 0)$  and  $(z_{2n-1}, z_{2n}) = (z_{-1}, z_0)$  for  $n \in \mathbb{N}_0$ . Therefore,

$$\dots, (0, 0, \varphi), (0, 0, \psi), (0, 0, \varphi), (0, 0, \psi), \dots$$

is a period two solution of system (4) with  $z_{-2}, z_0 = \varphi > 0$  and  $z_{-1} = \psi > 0$ . Furthermore, from (16), we have

$$\begin{aligned} x_{2n+1} - x_{2n-1} &= -\frac{x_{2n-1}y_{2n-2}^p}{1+y_{2n-2}^p} \leq 0, \\ y_{2n+1} - y_{2n-1} &= -\frac{y_{2n-1}z_{2n-2}^q}{1+z_{2n-2}^q} \leq 0, \\ z_{2n+1} - z_{2n-1} &= -\frac{z_{2n-1}x_{2n-2}^r}{1+x_{2n-2}^r} \leq 0 \end{aligned} \quad (18)$$

and

$$\begin{aligned} x_{2n+2} - x_{2n} &= -\frac{x_{2n}y_{2n-1}^p}{1+y_{2n-1}^p} \leq 0, \\ y_{2n+2} - y_{2n} &= -\frac{y_{2n}z_{2n-1}^q}{1+z_{2n-1}^q} \leq 0, \\ z_{2n+2} - z_{2n} &= -\frac{z_{2n}x_{2n-1}^r}{1+x_{2n-1}^r} \leq 0. \end{aligned} \tag{19}$$

From (18) and (19), we get

$$\begin{aligned} x_{2n+1} &\leq x_{2n-1}, & y_{2n+1} &\leq y_{2n-1}, & z_{2n+1} &\leq z_{2n-1} & \text{and} \\ x_{2n+2} &\leq x_{2n}, & y_{2n+2} &\leq y_{2n}, & z_{2n+2} &\leq z_{2n}. \end{aligned}$$

That is, the sequences  $(x_{2n-1}, y_{2n-1}, z_{2n-1})$  and  $(x_{2n}, y_{2n}, z_{2n})$  are non-increasing. Hence, while the odd-index terms tend to one periodic point and the even-index terms tend to another periodic point. This completes the proof.  $\square$

## 6. NUMERICAL EXAMPLES

In this section, we support our theoretical results related to system (4) with some numerical examples.

**Example 8.** *In the following figures, we illustrate the solution which corresponds to the initial conditions  $x_{-2} = 0.1$ ,  $x_{-1} = 1.2$ ,  $x_0 = 0.17$ ,  $y_{-2} = 0.11$ ,  $y_{-1} = 1.12$ ,  $y_0 = 2.17$ ,  $z_{-2} = 3.1$ ,  $z_{-1} = 2.12$ ,  $z_0 = 0.1$  and  $p = 3$ ,  $q = 2$ ,  $r = 4$  of (4) for difference values of the parameters  $a$ ,  $b$ ,  $c$ .*

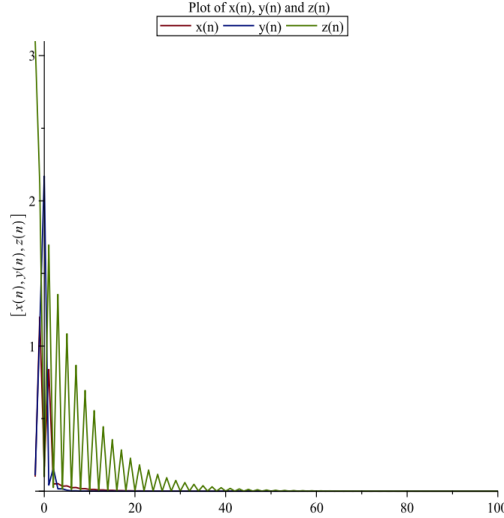


FIGURE 1.  $a = 0.7$ ,  $b = 0.4$ ,  $c = 0.8$  and  $p = 3$ ,  $q = 2$ ,  $r = 4$

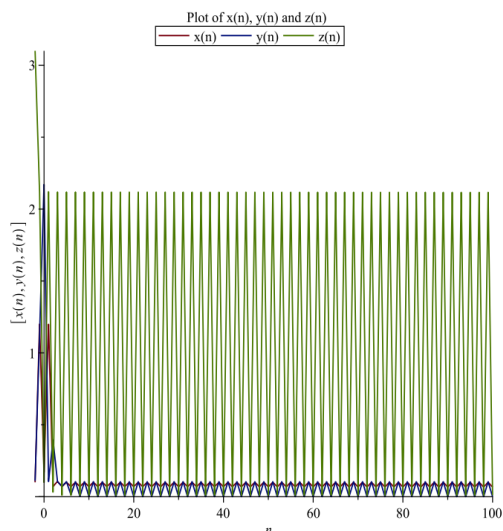


FIGURE 2.  $a = b = c = 1$  and  $p = 3, q = 2, r = 4$

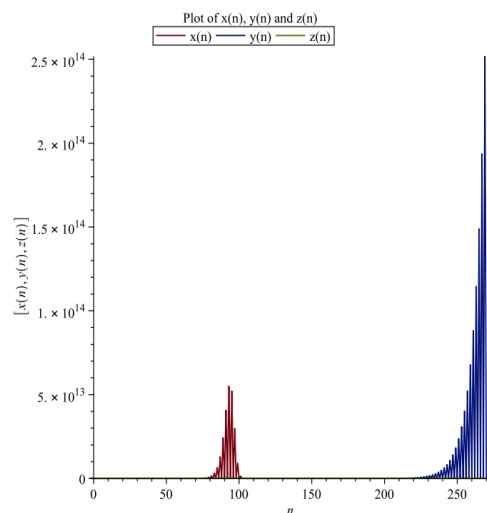
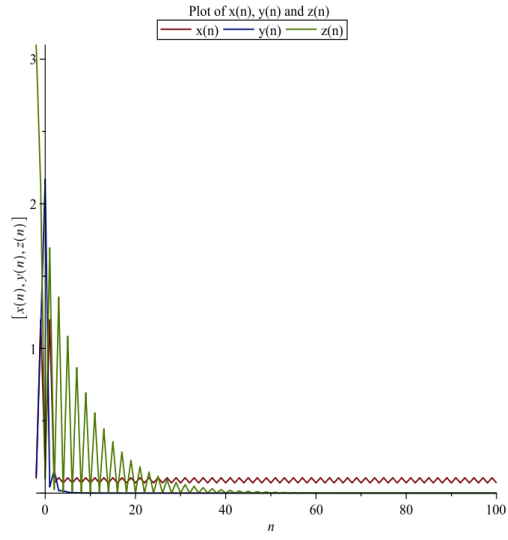
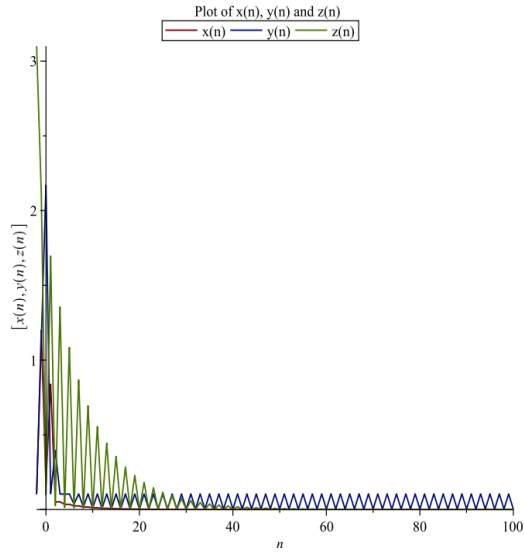


FIGURE 3.  $a = 2.1, b = 1.3, c = 1.1$  and  $p = 3, q = 2, r = 4$

FIGURE 4.  $a = 1$ ,  $b = 0.4$ ,  $c = 0.8$  and  $p = 3$ ,  $q = 2$ ,  $r = 4$ FIGURE 5.  $a = 0.7$ ,  $b = 1$ ,  $c = 0.8$  and  $p = 3$ ,  $q = 2$ ,  $r = 4$

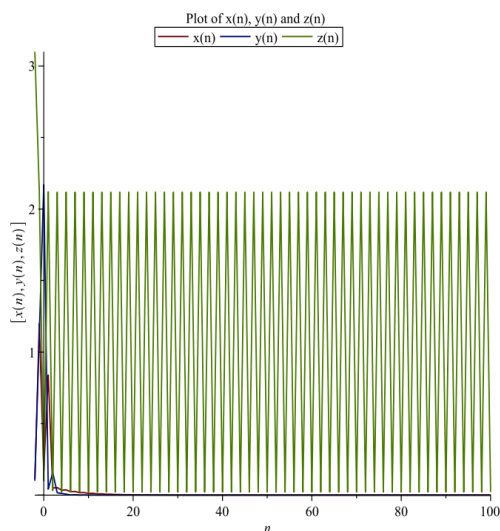


FIGURE 6.  $a = 0.7$ ,  $b = 0.4$ ,  $c = 1$  and  $p = 3$ ,  $q = 2$ ,  $r = 4$

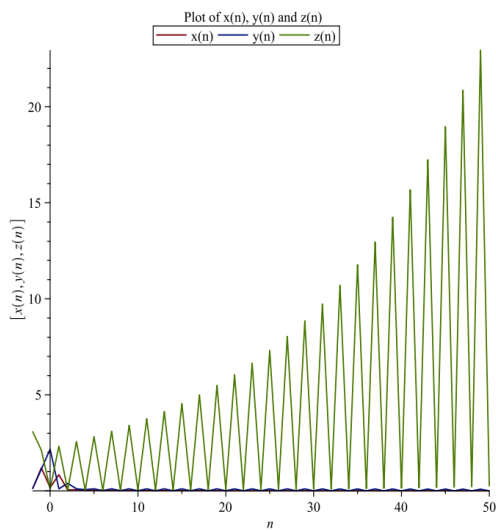


FIGURE 7.  $a = 0.7$ ,  $b = 1$ ,  $c = 1.1$  and  $p = 3$ ,  $q = 2$ ,  $r = 4$

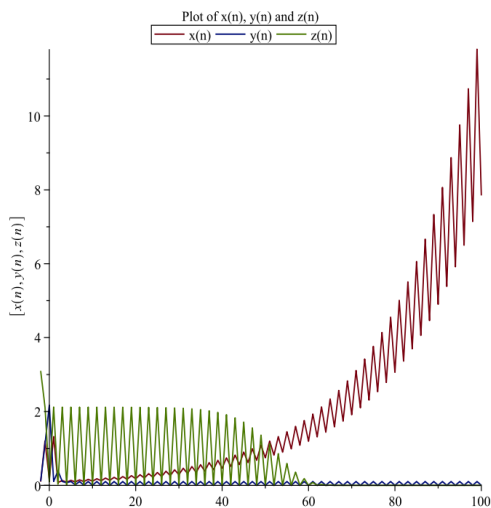


FIGURE 8.  $a = 1.1$ ,  $b = 1$ ,  $c = 1$  and  $p = 3$ ,  $q = 2$ ,  $r = 4$

#### REFERENCES

- [1] Din, Q., Qureshi M. N. and Khan, A. Q., Dynamics of a fourth-order system of rational difference equations, *Advances in Difference Equations*, (2012), 2012:215.
- [2] Din, Q. and Elsayed, E. M., Stability analysis of a discrete ecological model, *Computational Ecology and Software*, 4(2)(2014), 89-103.
- [3] Din, Q., Ibrahim, T. F. and Khan, A. Q., Behavior of a competitive system of second-order difference equations, *The Scientific World Journal*, vol. 2014, Article ID 283982, 9 pages.
- [4] Elabbasy, E. M., El-Metwally, H. and Elsayed, E. M., Some properties and expressions of solutions for a class of nonlinear difference equation, *Utilitas Mathematica*, 87(2012), 93-110.
- [5] El-Metwally, H. and Elsayed, E. M., Solution and behavior of a third rational difference equation, *Utilitas Mathematica*, 88(2012), 27-42.
- [6] El-Metwally, H., Yalcinkaya, I. and Cinar, C., Global stability of an economic model, *Utilitas Mathematica*, 95(2014), 235-244.
- [7] El-Owaidy, H. M., Ahmed, A. M. and Youssef, A. M., The dynamics of the recursive sequence  $x_{n+1} = \frac{\alpha x_{n-1}}{\beta + \gamma x_{n-2}^p}$ , *Applied Mathematics Letters*, 18(9)(2005), 1013-1018.
- [8] Elaydi, S., An introduction to difference equations, third edition, undergraduate texts in mathematics, *Springer*, New York, 1999.
- [9] Elsayed, E. M., El-Dessoky, M. M. and Alotaibi, A., On the solutions of a general system of difference equations, *Discrete Dynamics in Nature and Society*, Article ID 892571, (2012).
- [10] Gumus, M. and Soykan, Y., Global character of a six dimensional nonlinear system of difference equations, *Discrete Dynamics in Nature and Society*, vol. 2016, Article ID 6842521, 7 pages.
- [11] Kocic, V. L., and Ladas, G., Global behavior of nonlinear difference equations of higher order with applications, *Kluwer Academic*, Dordrecht, 1993.

- [12] Kulenović, M. R. S. and Nurkanović, Z., Global behavior of a three-dimensional linear fractional system of difference equations, *Journal of Mathematical Analysis and Applications*, 310(2)(2005), 673-689.
- [13] Kurbanli, A. S., Cinar, C. and Yalcinkaya, I., On the behavior of positive solutions of the system of rational difference equations  $x_{n+1} = x_{n-1}/(y_n x_{n-1} + 1)$ ,  $y_{n+1} = y_{n-1}/(x_n y_{n-1} + 1)$ , *Mathematical and Computer Modelling*, 53(5-6)(2011), 1261-1267.
- [14] Kurbanli, A. S., On the behavior of solutions of the system of rational difference equations  $x_{n+1} = x_{n-1}/(y_n x_{n-1} - 1)$ ,  $y_{n+1} = y_{n-1}/(x_n y_{n-1} - 1)$ ,  $z_{n+1} = 1/(y_n z_n)$ , *Advances in Difference Equations*, 2011: 40.
- [15] Kurbanli, A. S., On the behavior of solutions of the system of rational difference equations  $x_{n+1} = x_{n-1}/(y_n x_{n-1} - 1)$ ,  $y_{n+1} = y_{n-1}/(x_n y_{n-1} - 1)$ ,  $z_{n+1} = z_{n-1}/(y_n z_{n-1} - 1)$ , *Discrete Dynamics in Nature and Society*, vol. 2011, Article ID 932362, 12 pages.
- [16] Ozkan, O. and Kurbanli, A. S., On a system of difference equations, *Discrete Dynamics in Nature and Society*, vol. 2013, Article ID 970316, 7 pages.
- [17] Papaschinopoulos, G., Ellina, G. and Papadopoulos, K. B., Asymptotic behavior of the positive solutions of an exponential type system of difference equations, *Applied Mathematics and Computation*, 245(2014), 181-190.
- [18] Papaschinopoulos, G., Psarros, N. and Papadopoulos, K. B., On a cyclic system of  $m$  difference equations having exponential terms, *Electronic Journal of Qualitative Theory of Differential Equations*, 5(2015), 1-13.
- [19] Taskara, N., Uslu, K. and Tollu, D.T., The periodicity and solutions of the rational difference equation with periodic coefficients, *Computers & Mathematics with Applications*, 62(2011), 1807-1813.
- [20] Taskara, N., Tollu, D. T. and Yazlik, Y., Solutions of rational difference system of order three in terms of Padovan numbers, *Journal of Advanced Research in Applied Mathematics*, 7(3)(2015), 18-29.
- [21] Tollu, D. T., Yazlik Y. and Taskara, N., On the solutions of two special types of Riccati difference equation via Fibonacci numbers, *Advances in Difference Equations*, 2013:174, (2013).
- [22] Tollu, D. T., Yazlik Y. and Taskara, N., On fourteen solvable systems of difference equations, *Applied Mathematics and Computation*, 233(2014), 310-319.
- [23] Tollu, D. T., Yazlik Y. and Taskara, N., The solutions of four Riccati difference equations associated with Fibonacci numbers, *Balkan Journal Of Mathematics*, 2(2014), 163-172.
- [24] Touafek, N. and Elsayed, E. M., On the periodicity of some systems of nonlinear difference equations, *Bulletin Mathématique de la Société des Sciences Mathématiques de Roumanie*, 55(103), No: 2, (2012), 217-224.
- [25] Van Khuong, V. and Nam Phong, M., On a system of two difference equations of exponential form, *International Journal of Difference Equations*, 8(2)(2013), 215-223.
- [26] Yalcinkaya, I., Cinar, C. and Simsek, D., Global asymptotic stability of a system of difference equations, *Applicable Analysis*, 87(6)(2008), 689-699.
- [27] Yalcinkaya, I., On the global asymptotic stability of a second-order system of difference equations, *Discrete Dynamics in Nature and Society*, vol. 2008, Article ID 860152, 12 pages.
- [28] Yalcinkaya, I., Cinar, C. and Atalay, M., On the solutions of systems of difference equations, *Advances in Difference Equations*, 9(2008), Article ID 143943.
- [29] Yazlik, Y., Tollu, D. T. and Taskara, N., On the solutions of difference equation systems with Padovan numbers, *Applied Mathematics*, 4(2013), 15-20.
- [30] Yazlik, Y., On the solutions and behavior of rational difference equations, *Journal of Computational Analysis and Applications*, 17(3)(2014), 584-594.
- [31] Yazlik, Y., Elsayed, E. M. and Taskara, N., On the behaviour of the solutions of difference equation systems, *Journal of Computational Analysis and Applications*, 16(5)(2014), 932-941.

- [32] Yazlik, Y., Tollu, D. T. and Taskara, N., On the behaviour of solutions for some systems of difference equations, *Journal of Computational Analysis & Applications*, 18(1)(2015), 166-178.
- [33] Yazlik, Y., Tollu, D. T. and Taskara, N., On the solutions of a max-type difference equation system, *Mathematical Methods in the Applied Sciences*, 38(17)(2015), 4388–4410.
- [34] Yazlik, Y., Tollu, D. T. and Taskara, N., On the solutions of a three-dimensional system of difference equations, *Kuwait Journal of Science*, 43(1)( 2016), 95-111.

*Current address:* Durhasan Turgut Tollu: Necmettin Erbakan University, Faculty of Science, Department of Mathematics-Computer Sciences, Konya, Turkey

*E-mail address:* [dtollu@konya.edu.tr](mailto:dtollu@konya.edu.tr)

ORCID Address: <http://orcid.org/0000-0002-3313-8829>

*Current address:* İbrahim Yalçinkaya: Necmettin Erbakan University, Faculty of Science, Department of Mathematics-Computer Sciences, Konya, Turkey

*E-mail address:* [iyalcinkaya@konya.edu.tr](mailto:iyalcinkaya@konya.edu.tr)

ORCID Address: <http://orcid.org/0000-0003-4546-4493>